On L^2 norms of derivatives of orthogonal polynomials with respect to Sobolev inner products

Art B. Owen
Stanford University

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Abstract

For $\lambda \geqslant 0$, let $\langle f, g\rangle_\lambda := \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x + \lambda\int_{-1}^{1} f'(x)g'(x)\,\mathrm{d}x$ define an inner product for differentiable functions on $[-1, 1]$. For $n \geqslant 0$, let $S_n = S_n^\lambda$ be the orthogonal polynomials of degree n obtained by applying the Gram-Schmidt algorithm in this inner product to monomials, normalized so that $S_n(1) = 1$. Then the derivatives S_n' are given as explicit Legendre series, it is proved that $\arg\min_{n\geqslant 1} \int_{-1}^{1} S_n'(x)^2\,\mathrm{d}x / \int_{-1}^{1} S_n(x)^2\,\mathrm{d}x = 1$, and an expression is given for $\int_{-1}^{1} S_n'(x)S_{n'}'(x)\,\mathrm{d}x$.

1 Introduction

This work grew out of an effort to study effective dimension of some weighted Sobolev spaces used in quasi-Monte Carlo sampling. A first paper (Owen, 2014) obtained effective dimension values, by considering some Fourier series but the result was only applicable to integrands with periodic mixed partial derivatives. I looked at Legendre series as a way to remove the periodicity constraint and then looked at replacing the Legendre polynomials by orthogonal polynomials with respect to a Sobolev inner product on a finite interval. Ultimately, an approach based on Poincaré inequalities proved better and simpler. That is in Owen (2017). This paper reports some manipulations of orthogonal polynomials with respect to weighted Sobolev spaces that others might find useful.

2 Notation

We are interested in the orthogonal polynomials in a one dimensional Sobolev space. The field originates with Althammer (1962). Meijer (1996) gives a survey of orthogonal polynomials for Sobolev spaces including Schäfke (1972) which includes a key identity for this work.

The inner product we use is

$$\langle f, g\rangle_\lambda = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x + \lambda\int_{-1}^{1} f'(x)g'(x)\,\mathrm{d}x \tag{1}$$

for $\lambda \geqslant 0$. The case $\lambda = 0$ yields the classical Legendre polynomials. We use $\|f\|_\lambda$ for $\sqrt{\langle f, f\rangle_\lambda}$ and $\|f\|^2 = \int_{-1}^{1} f(x)^2\,\mathrm{d}x$.

Let $S_n = S_n^\lambda$ for integers $n \geqslant 0$ be orthogonal polynomials of degree n with respect to the inner product 1. They are normalized so that $S_n(1) = 1$, just like the Legendre polynomials that we denote by P_n.

Theorem 1. *For integers $n \geqslant 1$ and $\lambda \geqslant 0$,*

$$S_n = S_{n-2} + a_n(P_n - P_{n-2}), \quad \textit{where} \tag{2}$$

$$a_n = a_n(\lambda) = \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} \Big(\frac{\lambda}{4}\Big)^k \frac{1}{(2k)!}\frac{(n+2k-1)!}{(n-2k-1)!}, \tag{3}$$

and we use $S_{-1} = S_0 = P_{-1} = P_0 = 0$.

Proof. Schäfke (1972). □

If $\lambda = 0$, then we adopt the convention that $(\lambda/4)^0 = 1$, to include the ordinary Legendre polynomials via $a_n(0) = 1$.

Starting with $S_0 = P_0 = 1$ it can be shown from (2) that $S_1 = P_1 = x$ and $S_2 = P_2 = (3x^2 - 1)/2$. The first polynomial to depend on λ is $S_3 = x + a_3(\lambda)(P_3 - P_1)$, with $S_3(x) = x + (5/2)(1 + 3\lambda)(x^3 - x)$.

Proposition 1. *For $n \geqslant 1$ and $\lambda \geqslant 0$ define $a_n(\lambda)$ by (3). Then $a_{n+1}(\lambda) \geqslant a_n(\lambda) \geqslant a_1(\lambda) = 1$.*

Proof. The coefficient $(n + 2k - 1)!/(n - 2k - 1)!$ is non-decreasing in n. Also the terms in (3) are non-negative and their number is non-decreasing in n. Therefore $a_{n+1}(\lambda) \geqslant a_n(\lambda) \geqslant a_1(\lambda)$. Finally, $a_1(\lambda) = (\lambda/4)^0 = 1$. □

In studies of effective dimension, we are interested in finding the index of the 'least penalized' (nonconstant) orthogonal polynomial. This is

$$n^* = n^*(\lambda) = \arg\min_{n\geqslant 1} \frac{\|S_n\|_\lambda^2}{\|S_n\|^2} = \arg\min_{n\geqslant 1} \frac{\|S_n'\|^2}{\|S_n\|^2}.$$

For $\lambda = 0$, if we define the least penalized orthogonal polynomial to be $\arg\min_{n\geqslant 1} \|P_n'\|^2/\|P_n\|^2$ then $n^* = 1$ with a ratio of 3. We prove below that $n^*(\lambda) = 1$ for all $\lambda \geqslant 0$.

Proposition 2. *For $n \geqslant 1$, and $\lambda \geqslant 0$,*

$$S_n^\lambda = \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} a_{n-2k}(\lambda)(P_{n-2k} - P_{n-2k-2}). \tag{4}$$

Proof. If n is odd, then we apply the Schäfke formula (2) $(n+1)/2$ times to S_n yielding

$$S_n = \sum_{k=0}^{(n-1)/2} a_{n-2k}(P_{n-2k} - P_{n-2k-2}),$$

because $S_{-1} = 0$. If n is even, then we apply (2) $n/2 - 1$ times to yield

$$S_n = S_2 + \sum_{k=0}^{n/2-2} a_{n-2k}(P_{n-2k} - P_{n-2k-2}).$$

Now $S_2 = P_2 - P_0 = a_2(P_2 - P_0)$ as $a_2(\lambda) = 1$, producing a $k = n/2 - 1$ term for the sum above. Therefore both even and odd n can be written as in (4). □

Corollary 1. *For $n \geqslant 1$ and $\lambda \geqslant 0$, let $S_n = S_n^\lambda$. Then*

$$S_n' = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n - 4k - 1) a_{n-2k}(\lambda) P_{n-2k-1}. \tag{5}$$

Proof. Differentiate (4) term by term and apply a standard Legendre function identity: $P_{m+1}' - P_{m-1}' = (2m+1)P_n$, with $m = n - 2k - 1$. □

Proposition 3. *For $n \geqslant 1$ and $\lambda \geqslant 0$, let $S_n = S_n^\lambda$. Then*

$$\|S_n'\|^2 = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n-2k}^2 (2n - 4k - 1). \tag{6}$$

Proof. The Legendre functions are orthogonal with $\|P_n\|^2 = 2/(2n+1)$, and so

$$\|S_n'\|^2 = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n-2k}^2 \frac{(2n-4k-1)^2}{2(n-2k-1)+1} = 2 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n-2k}^2 (2n - 4k - 1),$$

where we have used $\|P_n\|^2 = 2/(2n+1)$. □

Proposition 4. *For $n \geqslant 1$ and $\lambda \geqslant 0$, let $S_n = S_n^\lambda$. Then $\|S_n\|^2$ equals*

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n-2k}^2 \left(\|P_{n-2k}\|^2 + \|P_{n-2k-2}\|^2 \right) \\ &- \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} a_{n-2k} a_{n-2k+2} \|P_{n-2k}\|^2 - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} a_{n-2k} a_{n-2k-2} \|P_{n-2k-2}\|^2. \end{aligned} \tag{7}$$

Proof. Write

$$\|S_n\|^2 = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \sum_{k'=0}^{\lfloor\frac{n-1}{2}\rfloor} a_{n-2k} a_{n-2k'} \big\langle (P_{n-2k} - P_{n-2k-2})(P_{n-2k'} - P_{n-2k'-2}) \big\rangle$$

and then collect up the nonzero terms. □

Proposition 5. *For $n \geqslant 1$ and $\lambda \geqslant 0$, let $S_n = S_n^\lambda$. Then*

$$\|S'_{n+2}\|^2 - \|S'_n\|^2 = (4n+6)a_{n+2}^2.$$

Proof. From (6), $\|S'_{n+2}\|^2 - \|S'_n\|^2$ equals

$$2 \sum_{k'=0}^{\lfloor\frac{n-1}{2}\rfloor+1} a_{n+2-2k'}^2 (2(n+2) - 4k' - 1) - 2 \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} a_{n-2k}^2 (2n - 4k - 1).$$

For $k \geqslant 0$, the k'th term in the second sum cancels the $k' = k+1$'st term in the first sum, leaving only the $k' = 0$ term. □

Proposition 6. *For $n \geqslant 1$ and $\lambda \geqslant 0$, let $S_n = S_n^\lambda$. Then*

$$\|S_{n+2}\|^2 - \|S_n\|^2 = a_{n+2}^2\big(\|P_{n+2}\|^2 + \|P_n\|^2\big) - a_n a_{n+2}\|P_n\|^2 - a_{n+2}a_n\|P_{n-2}\|^2.$$

Proof. The proof is similar to that of Proposition 6. Proposition 4 expresses $\|S_n\|^2$ through three sums. In the difference $\|S_{n+2}\|^2 - \|S_n\|^2$ all the terms cancel except for lowest index terms of S_{n+2}. □

Lemma 1. *For any $\lambda \geqslant 0$, and $n > 1$, $\|S'_n\|^2/\|S_n\|^2 > \|S'_1\|^2/\|S_1\|^2$, so $n^* \equiv \arg\min_{n\geqslant 1} \|S'_n\|^2/\|S_n\|^2 = 1$.*

Proof. First, $\|S'_1\|^2/\|S_1\|^2 = 3$ and $\|S'_2\|^2/\|S_2\|^2 = 15$. We will show that increasing n by two raises $\|S'_n\|^2$ by more than three times as much as it raises $\|S_n\|^2$. For $n \geqslant 1$,

$$\big(\|S'_{n+2}\|^2 - \|S'_n\|^2\big) - 3\big(\|S_{n+2}\|^2 - \|S_n\|^2\big) \geqslant a_{n+2}^2\big((4n+6) - \|P_{n+2}\|^2 - \|P_n\|^2\big),$$

by Propositions 5 and 6. Next, for $n \geqslant 1$

$$(4n+6) - \|P_{n+2}\|^2 - \|P_n\|^2 = (4n+6) - \frac{2}{2n+5} - \frac{2}{2n+1} > 0.$$

If n is odd, then $\|S'_n\|^2/\|S_n\|^2 > \|S'_1\|^2/\|S_1\|^2$ while if n is even, $\|S'_n\|^2/\|S_n\|^2 > \|S'_2\|^2/\|S_2\|^2 > \|S'_1\|^2/\|S_1\|^2$. □

Proposition 7. *For $n' \geqslant n \geqslant 0$,*

$$\int_{-1}^{1} S'_n(x) S'_{n'}(x)\,\mathrm{d}x = \begin{cases} \|S'_n\|^2, & n' - n \equiv 0 \mod 2 \\ 0, & n' - n = 1 \mod 2. \end{cases}$$

Proof. The polynomial S'_n is an odd function for even n and an even function for odd n, so the inner product vanishes if $n' - n \neq 0 \mod 2$. Otherwise via Corollary 1, we see that $S'_{n'} - S'_n$ is a linear combination of $P_{n'-2k'-1}$ for $n'-2k'-1 > n-2k-1$. These are orthogonal to S'_n and so the result follows. □

References

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