Control Variates with Quasi-Monte Carlo

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QMC + CV

$$I = \int_{(0,1)^s} f(x)dx, \quad s \text{ large}$$

Simple MC most widely used

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(x_i), \quad x_i \sim U(0,1)^s$$

Improvements

1. Quasi-Monte Carlo, replacing x_i
2. Control variates, using related known integrals

Try both

Obvious to do this, but some surprises:

1. CV coefficient can change
2. good variates also change

QMC

Use low discrepancy x_i instead of random

Main methods:

nets Niederreiter, Sobol', Faure, Xing . . .

lattices Sloan + Joe, Korobov, Hua, Wang, Fang . . .

Randomized QMC: allows error estimation, and some errors cancel Lemieux + L'Ecuyer, Hong + Hickernell, Cranley + Patterson, Owen, Matousek . . .

Sampling Methods

1. Monte Carlo: $n^{-1/2}$
2. Quasi-Monte Carlo: $n^{-1}(\log n)^{d-1}$, but no practical error estimate
3. Randomized Quasi-Monte Carlo: replication based error estimates, and $n^{-3/2}(\log n)^{(d-1)/2}$ (for nets)

Rates are asymptotic under mild conditions on f

Control Variates

Know $\int h_j(x)dx = \theta_j$ for $j = 1, \ldots, J$

For $\beta = (\beta_1, \ldots, \beta_J)^T$, let:

$$\hat{I}_\beta = \frac{1}{n}\sum_{i=1}^{n}\Big(f(x_i) - \beta^T(h(x_i) - \theta)\Big)$$

Easily $E(\hat{I}_\beta) = I$ any β

Simple example (more later)

$J = 1$,

$f =$ Arithmetic mean option value, e.g. Asian option

$h = h_1 =$ Geometric mean option value

Get $\theta = \int h(x)dx$ by Black Scholes

$\beta = \beta_1 = 1$ works well because $f - h$ has small variance

CV variance

$$\text{Var}(\hat{I}_\beta) = \sigma^2_\beta / n$$

$$\sigma^2_\beta = E\big([f(x_i) - I - \beta^T(h(x_i) - \theta)]^2\big),$$

Unknown optimal β

$$\beta_{\text{mc}} = \left(\int (h(x) - \theta)(h(x) - \theta)^T dx\right)^{-1} \times \int (h(x) - \theta) f(x) dx.$$

Of course $\sigma^2_{\text{mc}} \equiv \sigma^2_{\beta_{\text{mc}}} \le \sigma^2$ Assume $\sigma^2_{\text{mc}} > 0$

Sample value $\hat{\beta}$

$$\hat{\beta}_{\text{mc}} = \left(\sum_{i=1}^n (h(x_i) - \hat{H})(h(x_i) - \hat{H})^T\right)^{-1} \times \sum_{i=1}^n (h(x_i) - \hat{H}) f(x_i)$$

$$\hat{H} = \frac{1}{n} \sum_{i=1}^n h(x_i)$$

MC and CV

Take $\beta = \hat{\beta}$ (Sample least squares)

$\hat{I}_{\hat{\beta}}$ almost as good as $\hat{I}_{\beta_{\text{mc}}}$

$\hat{I}_{\hat{\beta}}$ has small bias

Control variates forgiving of mild errors $\hat{\beta} - \beta_{\text{mc}}$

Common variance estimate

$$\widehat{\text{Var}}(\hat{I}_{\hat{\beta}_{\text{mc}}}) = \frac{1}{n - J - 1} \sum_{i=1}^{n} \Big(f(x_i) - \hat{I} - \hat{\beta}'(h(x_i) - \hat{H}) \Big)^2$$

or use $1/n$ vs $1/(n - J - 1)$

RQMC and CV

$$\text{Var}_{\text{rqmc}}(\hat{I}_\beta) = \text{Var}_{\text{rqmc}}\left(\hat{I} - \sum_{j=1}^{J} \beta_j \hat{H}_j\right)$$

Unknown optimal β

$$\beta_{\text{rqmc}} = \text{Cov}_{\text{rqmc}}(\hat{H}, \hat{H})^{-1}\text{Cov}_{\text{rqmc}}(\hat{H}, \hat{I})$$

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(x_i) \quad \hat{H} = \frac{1}{n}\sum_{i=1}^{n} h(x_i)$$

Note:

β_{rqmc} need not be close to β_{mc}

β_{rqmc} depends on n

Sample least squares estimates β_{mc} not β_{rqmc}

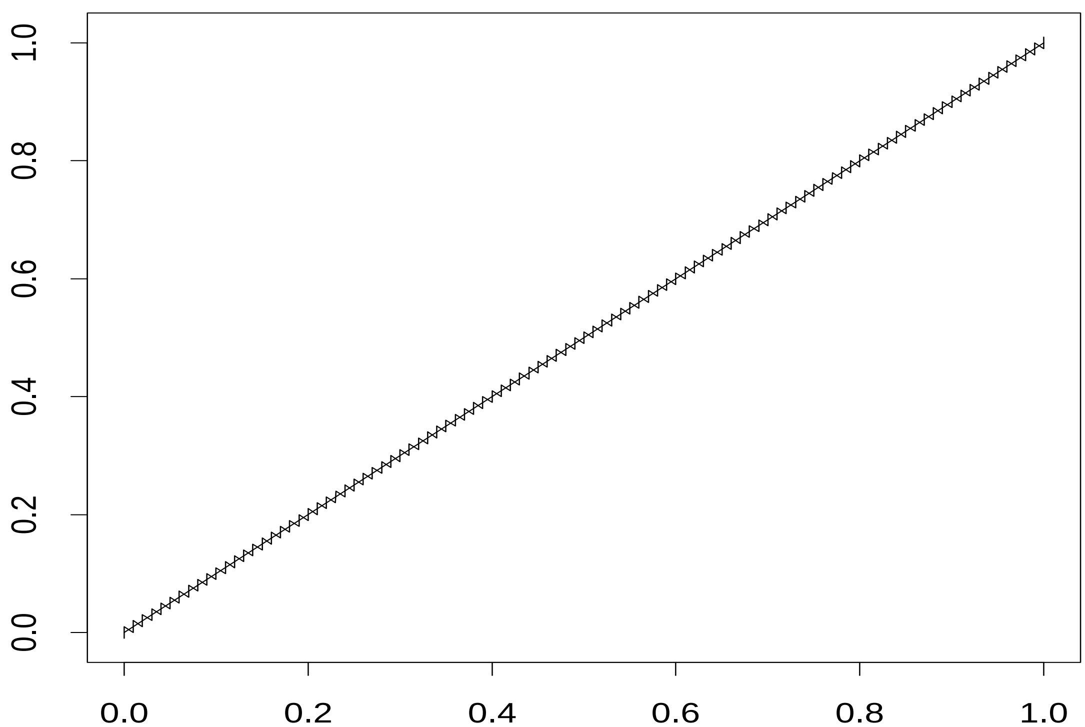
Good CV h should be correlated with f under RQMC, not necessarily under MC

Cautionary example

$$f(x) = (1 + 2\lfloor Mx \rfloor - Mx)/M$$

$$h(x) = x$$

$$M = 50$$



Cautionary ex. results

MC

$$\beta_{\text{mc}} = 1 - 2M^{-2} \doteq 1$$

$$\sigma^2_{\text{mc}} = 4\sigma^2(M^{-2} - M^{-4})$$

$$= \sigma^2/625.25, \qquad \text{for } M = 50$$

RQMC

Randomized $(0,1,1)$-net base $b = n$

IE n strata, $x_i \sim U[(i-1)/n, i/n)$ with $n = M$

$$\text{Var}_{\text{rqmc}}(\hat{I}) \doteq \frac{1}{12n^3}$$

$$\text{Var}_{\text{rqmc}}(\hat{I}_{\beta_{\text{mc}}}) \doteq \frac{2}{12n^3}$$

$$\beta_{\text{rqmc}} = -1$$

$$\text{Var}_{\text{rqmc}}(\hat{I}_{-1}) = 0$$

Stratification and CV

RQMC is something like stratification

For stratified sampling of $(0,1)$

$$\mathrm{Var}_{\text{strat}}\left(\frac{1}{n}\sum_{i=1}^{n} g(X_i)\right) \doteq \frac{1}{12n^3}\int_0^1 g'(x)^2 dx$$

Taking $g = f - h^T\beta$

$$\beta_{\text{strat}} \doteq \left(\int_0^1 h'(x)h'(x)^T dx\right)^{-1} \int h'(x)f'(x)dx$$

Correlation within strata counts more than correlation between strata

In the limit, a good CV has h' correlated with f' (uncentered)

Stratification of $(0,1)^s$

Asymptotic variance:

$$\frac{1}{12n^{1+2/s}}\int_{[0,1)^s}\Bigl\|\nabla\Bigl(f(x)-\beta^T h(x)\Bigr)\Bigr\|_2^2 dx$$

Asymptotic coefficient:

$$\Bigl(\int_{[0,1)^s}\nabla h\nabla h^T dx\Bigr)^{-1}\int_{[0,1)^s}\nabla h\nabla f\, dx$$

∇f has s components, one for each dimension of stratification

A good CV has gradient correlated with that of f

RQMC

$f(x) = f_G(x) + f_B(x)$

$\int f_G(x) f_B(x) dx = 0 \quad \int f_B(x) dx = 0$

RQMC “good” for f_G “bad” for f_B

Nets: f_B linear combination of indicators of small elementary intervals

Lattices: f_B part of f in dual lattice

Bad parts of h should correlate with those of f

$$\begin{aligned}
\hat{I}_\beta &= \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \beta^T(h(x_i) - \theta) \\
&= \frac{1}{n}\sum_{i=1}^{n} f_G(x_i) - \beta^T(h_G(x_i) - \theta) \\
&+ \frac{1}{n}\sum_{i=1}^{n} f_B(x_i) - \beta^T(h_B(x_i) - \theta) \\
&\doteq I + \frac{1}{n}\sum_{i=1}^{n} f_B(x_i) - \beta^T(h_B(x_i) - \theta)
\end{aligned}$$

RQMC

nets

For nets we can almost replace ∇ by s-fold partial derivative

$$\frac{\partial^s}{\partial x^{(1)} \partial x^{(2)} \dots \partial x^{(s)}}$$

lattices

No analogue yet. Depends on limiting behavior of dual lattice.

Small example

$f(x) = \sin(\pi(x^{(1)} + x^{(2)}))$ on $(0,1)^2$

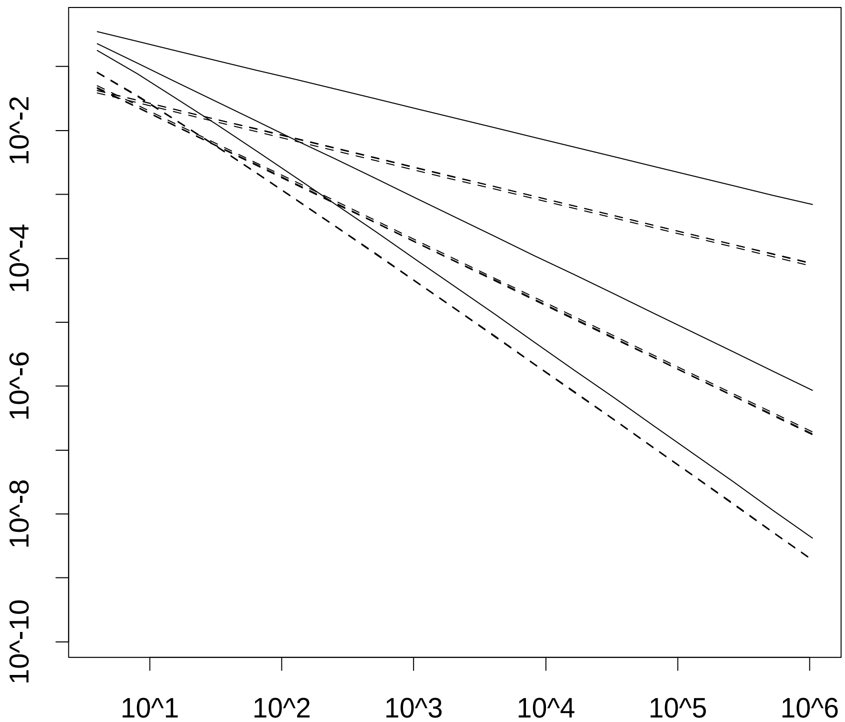
$h(x) = (x^{(1)} + x^{(2)} - 1)^3 - (x^{(1)} + x^{(2)} - 1)$

Asymptotic variance constants from low dimensional integrals

MC, Stratification, $(0, m, 2)$–nets

CV Coef $\rightarrow$ / Rate $\downarrow$	None 0	β_{mc} 2.675	β_{strat} 2.809	$\widetilde{\beta}_{\text{rnet}}$ 2.547	Gain
n^{-1}	0.5	0.0059	0.0072	0.0071	84.2
n^{-2}	0.825	0.0351	0.0333	0.0402	24.7
$n^{-3}\log(n)$	1.464	0.297	0.307	0.294	4.98

Asymptotic RMSE vs n



Dark lines for no CV

Dashed lines use CV

Top to bottom: MC, Strat, r-net

Equidistribution trumps CV

in this 2 dimensional example

Diminishing returns

For MC 10-fold variance reduction is a 10-fold time savings

For RQMC with Var$(\hat{I}) = O(n^{-2})$ a 10-fold variance reduction is a $10^{1/2} \doteq 3.16$-fold time saving

For RQMC with Var$(\hat{I}) = O(n^{-3})$ a 10-fold variance reduction is a $10^{1/3} \doteq 2.15$-fold time saving

Replication for β_{rqmc}

Run R independent replicates of RQMC with $\tilde{n}$ points

Get $\hat{I}_r$, $\hat{H}_r$ $r = 1, \dots, R$

RQMC is $R = 1$ with $\tilde{n} = n$

MC is $R = n$ with $\tilde{n} = 1$

Replication estimate

$$\hat{I}_\bullet = \frac{1}{R}\sum_{r=1}^{R}\hat{I}_r, \qquad \hat{H}_\bullet = \frac{1}{R}\sum_{r=1}^{R}\hat{H}_r$$

$$\hat{I}_{\hat{\beta}} = \hat{I}_\bullet - \hat{\beta}^T(\hat{H}_\bullet - \theta)$$

$$\hat{\beta} = \Bigl(\sum_{r=1}^{R}(\hat{H}_r - \hat{H}_\bullet)(\hat{H}_r - \hat{H}_\bullet)^T\Bigr)^{-1} \times \Bigl(\sum_{r=1}^{R}(\hat{H}_r - \hat{H}_\bullet)(\hat{I}_r - \hat{I}_\bullet)\Bigr).$$

Estimates β_{rqmc} for $\tilde{n} < n$ points

Error estimates

Minimize

$$\mathrm{SS}(\beta_0, \beta) = \sum_{r=1}^{R} (\hat{I}_r - \beta_0 - \hat{H}_r^T \beta)^2$$

with $\beta_0 = \hat{I}_{\hat{\beta}} + \hat{\beta}^T \theta$ and $\beta = \hat{\beta}$.

$\widehat{\mathrm{Var}}(\hat{\beta}) = \mathrm{SS}(\hat{I}_{\hat{\beta}}, \hat{\beta}) / (R(R - J - 1))$.

Internal replicates

Split $x_1, \dots, x_n$ into R consecutive blocks of $\tilde{n}$ points

A scrambled (λ, t, m, s)-net in base b splits into λb^r scrambled $(t, m-r, s)$-nets

Random shifted lattice rules with $n = b^m$ points split into $R = b^{m-\tilde{m}}$ such with $b^{\tilde{m}}$ points

Conservativism

Internal replicates tend to be negatively correlated

This deflates true error

and inflates error estimate

Aliasing

complicates internal replication of randomly shifted lattices

Picking R

$$\text{Var}_{\text{rqmc}}(\hat{I}_{\hat{\beta}}) = \frac{1}{R}\text{Var}_{\text{rqmc}}(\hat{I}_{\beta_{\text{rqmc},\tilde{n}}})(1 + O_p(R^{-1}))$$

The $1 + O_p(R^{-1})$ comes from errors in $\hat{\beta}$

Probably best to have bounded (or very slowly growing) R as $n \to \infty$

$$\begin{aligned}\text{Var}_{\text{rqmc}}(\hat{I}_{\beta_{\text{rqmc},\tilde{n}}}) &= O(\tilde{n}^{-2+\epsilon}), \quad \text{implies} \\ \text{Var}(\hat{I}_{\hat{\beta}}) &= O(\tilde{n}^{-2+\epsilon}R^{-1}) \\ &= O(R^{1-\epsilon}n^{-2+\epsilon})\end{aligned}$$

We probably need R over 5 or $J + 5$

Asian option example

Asset worth $S(t)$ at time t

Option pays $\max(0,(1/s)\sum_{i=1}^{s} S(t_i)-K)$ at T

$t_i = iT/s$

Geometric B.M. for $S(t_i)$

$$S(0)\exp\left[(r-\sigma^2/2)t_i + \sigma\sqrt{T/s}\sum_{j=1}^{i}\Phi^{-1}(x^{(j)})\right]$$

Discounted value:

$$f(x) = e^{-rT}\max\left(0, \frac{1}{s}\sum_{i=1}^{s} S(t_i, x) - K\right).$$

Control variate:

$$h_1(x) = e^{-rT}\max\left(0, \prod_{i=1}^{s} S(t_i, x)^{1/s} - K\right).$$

Parameters

Initial price	$S(0) = 100$
Strike price	$K = 120$
Risk-free interest	$r = 0.05$
Expiration	$T = 1$ year
Sampling	$s = 16$ times
Volatility	$\sigma = 0.3$

Out of money option.

Has about 0.17 probability of paying.

Truncation for BVHK

$f - A$ and $h - G$ are bounded, where

$$A(x) = e^{-rT}\left(\frac{1}{s}\sum_{i=1}^{s} S(t_i, x) - K\right)$$

$$G(x) = e^{-rT}\left(\prod_{i=1}^{s} S(t_i, x)^{1/s} - K\right)$$

Both A and G have known integrals

Maybe QMC works better for $f - A$

with or without the CV $h - G$

MC could be worse because $f - A$

can have large variance for out-of-money option

CV strategies, for MC

Name	*Estimate*
MC$_0$	$\hat{I}(f)$
MC$_1$	$\hat{I}(f - \beta_1 h_1) + \beta_1 I(h_1)$
MC$_3$	$\hat{I}(f - \beta_2 h_1 - \beta_3 A - \beta_4 G)$
	$+\beta_2 I(h_1) + \beta_3 I(A) + \beta_4 I(G)$
MC$_B$	$\hat{I}(f - A) + I(A)$
MC$_{BB}$	$\hat{I}(f - A - \beta_5(h_1 - G))$
	$+I(A) + \beta_5 I(h_1 - G)$

Same CV strategies used with RQMC

QMC$^{(2)}$: $R = 85$ reps of $(0, 2, 16)$-net in base 17

QMC$^{(3)}$: $R = 5$ reps of $(0, 3, 16)$-net in base 17
uses est of β_{mc}

Of course: MC has $n = 5 \times 17^3 = 24565$

Estimated coefficients

Coef	Of	In	MC	QMC$^{(2)}$	QMC$^{(3)}$
β_1	h	1	1.10	1.08	1.10
β_2	h	3	1.04	1.01	1.04
β_3	A	3	0.534	1.33	0.519
β_4	G	3	-0.525	-1.37	-0.510
β_5	$h - g$	BB	0.988	1.03	0.987

Remarks

1. Arith mean $>$ Geom mean, so $\beta_1 > 1$
2. A–coef plus G–coef $\doteq 0$
3. Only QMC$^{(2)}$ estimates β_{rqmc}

Estimated RMSE

	MC	$\mathrm{QMC}^{(2)}$	$\mathrm{QMC}^{(3)}$
0	4.41e−2	2.05e−2	3.70e−3
1	2.99e−3	2.16e−3	1.34e−3
3	2.08e−3	1.48e−3	1.04e−3
B	9.05e−2	1.69e−2	2.94e−3
BB	2.81e−3	1.52e−3	7.35e−4

Remarks

1. $\mathrm{QMC}^{(3)}$ beats $\mathrm{QMC}^{(2)}$ beats MC (always)
2. Methods that don't use CV do badly
3. MC_3 is best MC (as it should be)
4. $\mathrm{QMC}_3^{(2)}$ is best $\mathrm{QMC}^{(2)}$ (barely)
5. MC_1 almost as good as $\mathrm{QMC}_0^{(3)}$
6. $\mathrm{QMC}_{BB}^{(3)}$ best overall

Efficiency gains

CV gives more improvement than QMC

142 in replacing MC_0 by $\text{QMC}_0^{(3)}$

217 in replacing MC_0 by MC_1

450 in replacing MC_0 by MC_3

Synergy

3600 in replacing MC_0 by $\text{QMC}_{BB}^{(3)}$

BVHK

yields small gains

Conclusions

CV coefficients depend on sampling method

Effective variates can also depend on sampling method

QMC with sub-optimal CV coefficients worked well