Empirical Likelihood

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Thanks to

- University of Ottawa
- Fields Institute
- Mayer Alvo
- Jon Rao

This talk

- based on the book “Empirical Likelihood” (2001)
- starts with central topics, spirals out, ends with challenges

Empirical likelihood provides:

- **likelihood** methods for inference, especially
  - tests, and
  - confidence regions,
- **without** assuming a parametric model for data
- **competitive** power even when parametric model holds

Parametric likelihoods

Data have *known* distribution f_θ with *unknown parameter* θ

$$\Pr(X_1 = x_1, \ldots, X_n = x_n) = f(x_1, \ldots, x_n; \theta)$$
$$\Pr(x_1 \leq X_1 \leq x_1 + \Delta, \ldots, x_n \leq X_n \leq x_n + \Delta) \propto f(x_1, \ldots, x_n; \theta)$$

$f(\cdots; \cdot)$ known, $\theta \in \Theta \subseteq \mathbb{R}^p$ unknown

Likelihood function

$$L(\theta) = L(\theta; x_1, \ldots, x_n) = f(x_1, \ldots, x_n; \theta)$$

"Chance, under θ, of getting the data we did get"

Likelihood examples

$$X_i \sim \text{Poi}(\theta), \quad \theta \geq 0$$

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}$$

$$Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2) \quad x_i \text{ fixed}$$

$$L(\beta_0, \beta_1, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2}$$

Likelihood inference

Maximum likelihood estimate

$$\hat{\theta} = \arg\max_{\theta} L(\theta; x_1, \ldots, x_n)$$

Likelihood ratio inferences

$$-2\log(L(\theta_0)/L(\hat{\theta})) \to \chi^2_{(q)} \qquad \text{Wilks}$$

Typically . . . Neyman-Pearson, Cramer-Rao, . . .

1. $\hat{\theta}$ asymptotically normal
2. $\hat{\theta}$ asymptotically efficient
3. Likelihood ratio tests powerful
4. Likelihood ratio confidence regions small

Other likelihood advantages

- can model data distortion: bias, censoring, truncation
- can combine data from different sources
- can factor in prior information
- obey range constraints: MLE of correlation in $[-1, 1]$
- transformation invariance
- data determined shape for $\{\theta \mid L(\theta) \geq rL(\hat{\theta})\}$
- incorporates nuisance parameters

Unfortunately

We might not know a correct $f(\cdots;\theta)$

No reason to expect that new data belong to one of our favorite families

Wrong models sometimes work (e.g. Normal mean via CLT) and sometimes fail (e.g. Normal variance)

Also,

Usually easy to compute $L(\theta)$, but . . .

Sometimes hard to find $\hat{\theta}$

Sometimes hard to compute $\max_{\theta_2} L((\theta_1, \theta_2))$ (Profile likelihood)

Nonparametric methods

Assume only $X_i \sim F$ where

- F is continuous, or,
- F is symmetric, or,
- F has a monotone density, or,
- $\cdots$ other believable, but big, family

Nonparametric usually means infinite dimensional parameter

Sometimes lose power (e.g. sign test), sometimes not

Nonparametric maximum likelihood

For X_i IID from F, $\quad L(F) = \prod_{i=1}^{n} F(\{x_i\})$

The NPMLE is $\quad \widehat{F} = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$

where δ_x is a point mass at x

Kiefer and Wolfowitz, 1956

Proof

Distinct values z_j appear n_j times in sample, $j = 1, \ldots, m$

Let $F(\{z_j\}) = p_j \geq 0$ and $\widehat{F}(\{z_j\}) = \hat{p}_j = n_j/n$ with some $p_j \neq \hat{p}_j$

$$\begin{aligned}
\log\left(\frac{L(F)}{L(\widehat{F})}\right) &= \sum_{j=1}^{m} n_j \log\left(\frac{p_j}{\hat{p}_j}\right) \\
&= n \sum_{j=1}^{m} \hat{p}_j \log\left(\frac{p_j}{\hat{p}_j}\right) \\
&< n \sum_{j=1}^{m} \hat{p}_j \left(\frac{p_j}{\hat{p}_j} - 1\right) \\
&= 0. \quad \Box
\end{aligned}$$

Other NPMLEs

Kaplan-Meier	Right censored survival times
Lynden-Bell	Left truncated star brightness
Hartley-Rao	Sample survey data
Grenander	Monotone density for actuarial data

Nonparametric likelihood ratios

Likelihood ratio: $R(F) = L(F)/L(\widehat{F})$

Confidence region: $\{T(F) \mid R(F) \geq r\}$

Profile likelihood: $\mathcal{R}(\theta) = \sup\{R(F) \mid T(F) = \theta\}$

Confidence region: $\{\theta \mid \mathcal{R}(\theta) \geq r\}$

In parametric setting, $-2\log(r) = \chi^{2,1-\alpha}_{(q)}$

Suppose there are no ties

Let $w_i = F(\{x_i\}) \quad w_i \geq 0 \quad \sum_{i=1}^n w_i \leq 1$

$$L(F) = \prod_{i=1}^n w_i \quad L(\widehat{F}) = \prod_{i=1}^n 1/n \quad R(F) = \prod_{i=1}^n n w_i$$

$$\mathcal{R}(\theta) = \sup\Big\{\prod_{i=1}^n n w_i \mid T(F) = \theta\Big\}$$

If there are ties . . .

$$L(F) \to L(F) \times \prod_j n_j^{n_j}$$ and, $$L(\widehat{F}) \to L(\widehat{F}) \times \prod_j n_j^{n_j}$$

R and $\mathcal{R}$ unchanged

For the mean

$T(F) = \int x dF(x)$, $x \in \mathbb{R}^d$

$T(\widehat{F}) = \frac{1}{n} \sum_{i=1}^{n} x_i$

We get $\{T(F) \mid R(F) \geq \epsilon\} = \mathbb{R}^d, \quad \forall r < 1$

Let $F_{\epsilon,x} = (1-\epsilon)\widehat{F} + \epsilon\delta_x$

For any $r < 1$,

$R(F_{\epsilon,x}) = \frac{L((1-\epsilon)\widehat{F}+\epsilon\delta_x)}{L(\widehat{F})} \geq (1-\epsilon)^n \geq r$ for small enough ϵ

Then let δ_x range over $\mathbb{R}^d$

Fix for the mean

Restrict to $F(\{x_1, \ldots, x_n\}) = 1$ i.e. $\sum_{i=1}^{n} w_i = 1$

Confidence region is

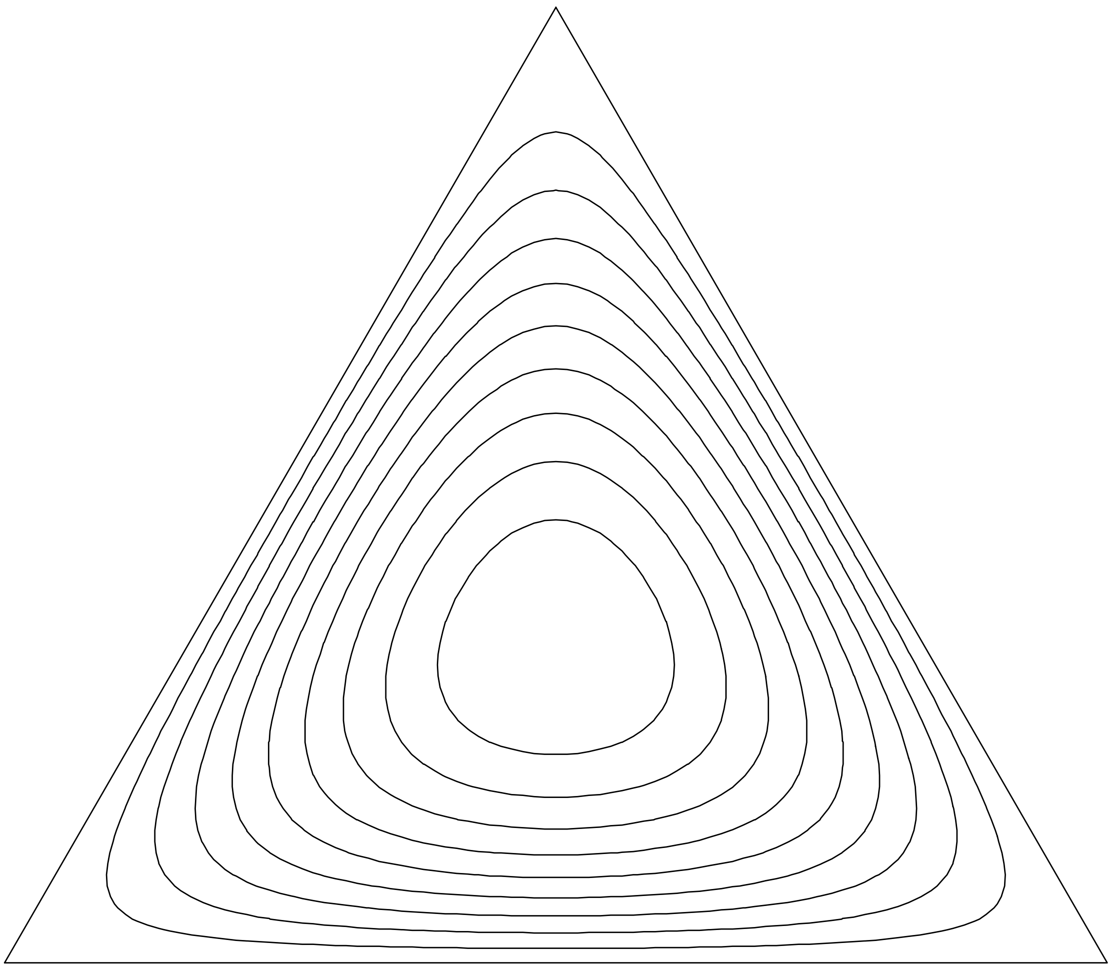
$$C_{r,n} = \left\{ \sum_{i=1}^{n} w_i x_i \mid w_i \geq 0, \sum_{i=1}^{n} w_i = 1, \prod_{i=1}^{n} n w_i > r \right\}$$

Profile likelihood

$$\mathcal{R}(\mu) = \sup \left\{ \prod_{i=1}^{n} n w_i \mid w_i > 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} w_i x_i = \mu \right\}$$

We have a multinomial on the n data points, hence $n - 1$ parameters

Multinomial likelihood for $n = 3$



MLE at center LR$= i/10, i = 0, \ldots, 9$

Empirical likelihood theorem

Suppose that $X_i \sim F_0$ are IID in $\mathbb{R}^d$

$\mu_0 = \int x dF_0(x)$

$V_0 = \int (x - \mu_0)(x - \mu_0)^T dF_0(x)$ finite

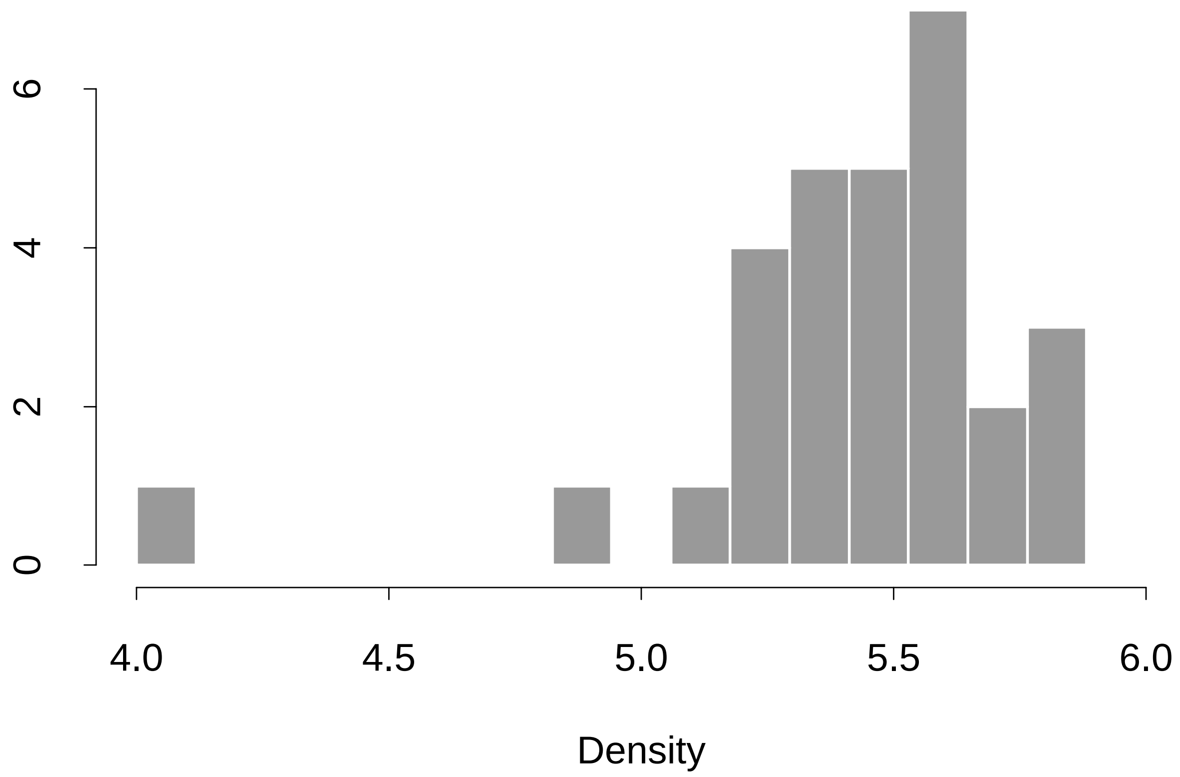
rank$(V_0) = q > 0$

Then as $n \longrightarrow \infty$

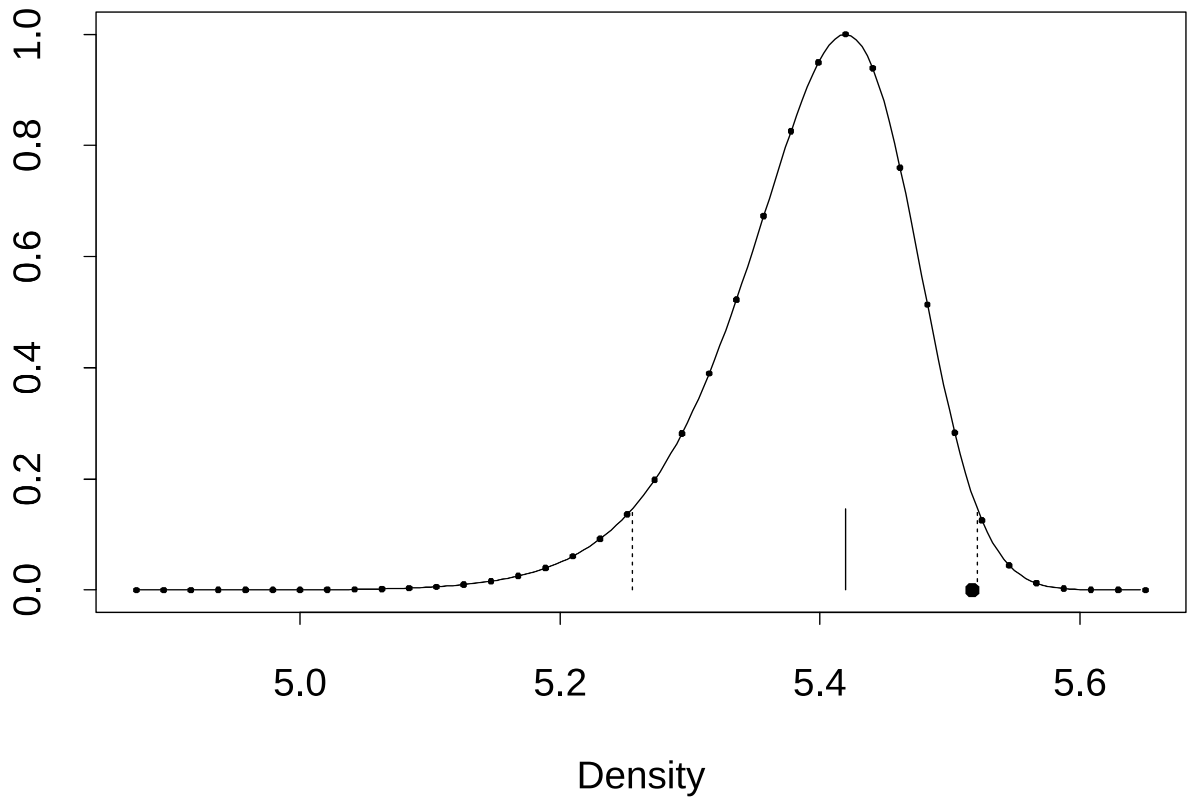
$$-2 \log \mathcal{R}(\mu_0) \to \chi^2_{(q)}$$

same as parametric limit

Cavendish's measurements of Earth's density



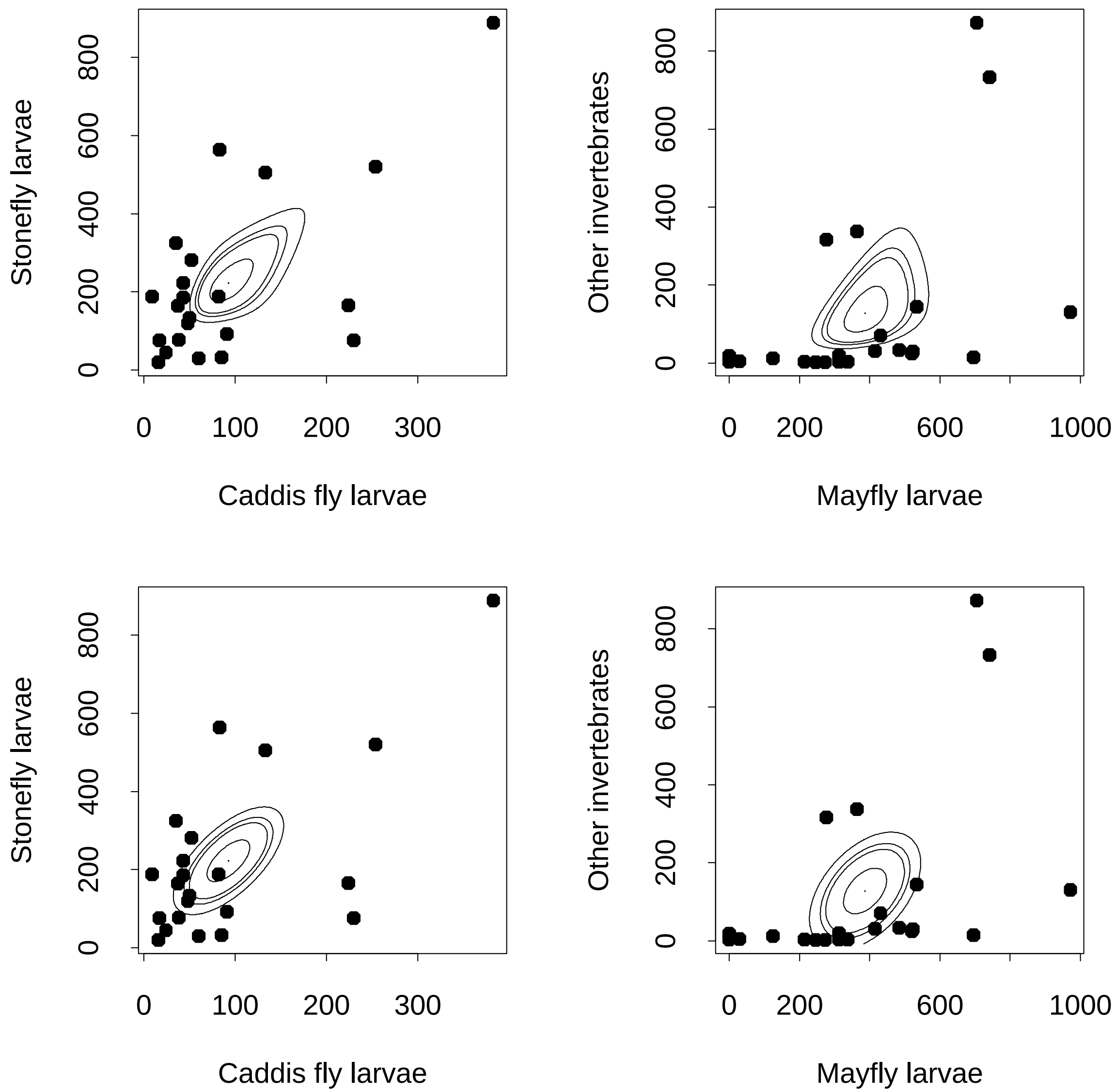
Profile empirical likelihood



Dipper, Cinclus cinclus

Eats larvae of Mayflies, Stoneflies, Caddis flies, other

Dipper diet means



Iles

Convex Hull

$$\mathcal{H} = \mathcal{H}(x_1, \ldots, x_n) = \left\{ \sum_{i=1}^{n} w_i x_i \mid w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}$$

$$\mu \notin \mathcal{H} \implies \log \mathcal{R}(\mu) = -\infty$$

If $\mu \in \mathcal{H}$ we get $\mathcal{R}(\mu)$ by Lagrange multipliers

Lagrange multipliers

$$G = \sum_{i=1}^{n} \log(nw_i) - n\lambda'\Big(\sum_{i=1}^{n} w_i(x_i - \mu)\Big) + \gamma\Big(\sum_{i=1}^{n} w_i - 1\Big)$$

$$\frac{\partial}{\partial w_i} G = \frac{1}{w_i} - n\lambda'(x_i - \mu) + \gamma = 0$$

$$\sum_i w_i \frac{\partial}{\partial w_i} G = n + \gamma = 0 \quad \Longrightarrow \quad \gamma = -n$$

Solving,

$$w_i = \frac{1}{n} \frac{1}{1 + \lambda'(x_i - \mu)}$$

Where $\lambda = \lambda(\mu)$ solves

$$0 = \sum_{i=1}^{n} \frac{x_i - \mu}{1 + \lambda'(x_i - \mu)}$$

Convex duality

$$\mathbb{L}(\lambda) \equiv -\sum_{i=1}^{n} \log(1 + \lambda'(x_i - \mu)) = \log R(F)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda} = -\sum_{i=1}^{n} \frac{x_i - \mu}{1 + \lambda'(x_i - \mu)}$$

Maximize $\log R$ or minimize $\mathbb{L}$

$$\frac{\partial^2 \mathbb{L}}{\partial \lambda \partial \lambda'} = \sum_{i=1}^{n} \frac{(x_i - \mu)(x_i - \mu)'}{(1 + \lambda'(x_i - \mu))^2}$$

$\mathbb{L}$ is convex and d dimensional $\Longrightarrow$ easy optimization

Sketch of ELT proof

WLOG $q = d$, and anticipate a small λ

$$0 = \frac{1}{n}\sum_{i=1}^{n} \frac{x_i - \mu}{1 + (x_i - \mu)'\lambda} \qquad 1/(1+\epsilon) = 1 - \epsilon + \epsilon^2 - \epsilon^3 \cdots$$

$$\doteq \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu) - (x_i - \mu)(x_i - \mu)'\lambda, \quad \text{so,}$$

$$\lambda \doteq S^{-1}(\bar{x} - \mu), \quad \text{where,}$$

$$S = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)(x_i - \mu)'$$

Left out: how $E(\|X\|^2) < \infty$ implies small $\lambda(\mu_0)$

Sketch continued

$$
\begin{aligned}
-2\log\prod_{i=1}^{n} nw_i &= -2\log\prod_{i=1}^{n}\frac{1}{1+\lambda'(x_i-\mu)} \\
&= 2\sum_{i=1}^{n}\log(1+\lambda'(x_i-\mu)) \qquad\qquad \log(1+\epsilon)=\epsilon-(1/2)\epsilon^2+\cdots \\
&\doteq 2\sum_{i=1}^{n}\Big(\lambda'(x_i-\mu)-\frac{1}{2}\lambda'(x_i-\mu)(x_i-\mu)'\lambda\Big) \\
&= n\Big(2\lambda'(\bar{x}-\mu)-\lambda' S\lambda\Big) \\
&= n\Big(2(\bar{x}-\mu)'S^{-1}(\bar{x}-\mu)-(\bar{x}-\mu)'S^{-1}SS^{-1}(\bar{x}-\mu)\Big) \\
&= n(\bar{x}-\mu)'S^{-1}(\bar{x}-\mu) \\
&\longrightarrow \chi^2_{(d)}
\end{aligned}
$$

Typical coverage errors

1. $\Pr(\mu_0 \in C_{r,n}) = 1 - \alpha + O\left(\frac{1}{n}\right)$ as $n \to \infty$
2. One-sided errors of $O\left(\frac{1}{\sqrt{n}}\right)$ cancel
3. Bartlett correction DiCiccio, Hall, Romano
   (a) replace $\chi^{2,1-\alpha}$ by $\left(1 + \frac{a}{n}\right)\chi^{2,1-\alpha}$ for carefully chosen a
   (b) get coverage errors $O\left(\frac{1}{n^2}\right)$
   (c) a does not depend on α
   (d) data based $\hat{a}$ gets same rate

same as for parametric likelihoods

Calibrating empirical likelihood

Plain $\chi^{2,1-\alpha}$	undercovers
$F^{1-\alpha}_{d,n-d}$	is a bit better
Bartlett correction	asymptotics slow to take hold
Bootstrap	seems to work best

Bootstrap calibration

Recipe

Sample X_i^* IID $\widehat{F}$

Get $-2\log \mathcal{R}(\bar{x}; x_1^*, \ldots, x_n^*)$

Repeat $B = 1000$ times

Use $1-\alpha$ sample quantile

Results

Regions get empirical likelihood shape and bootstrap size

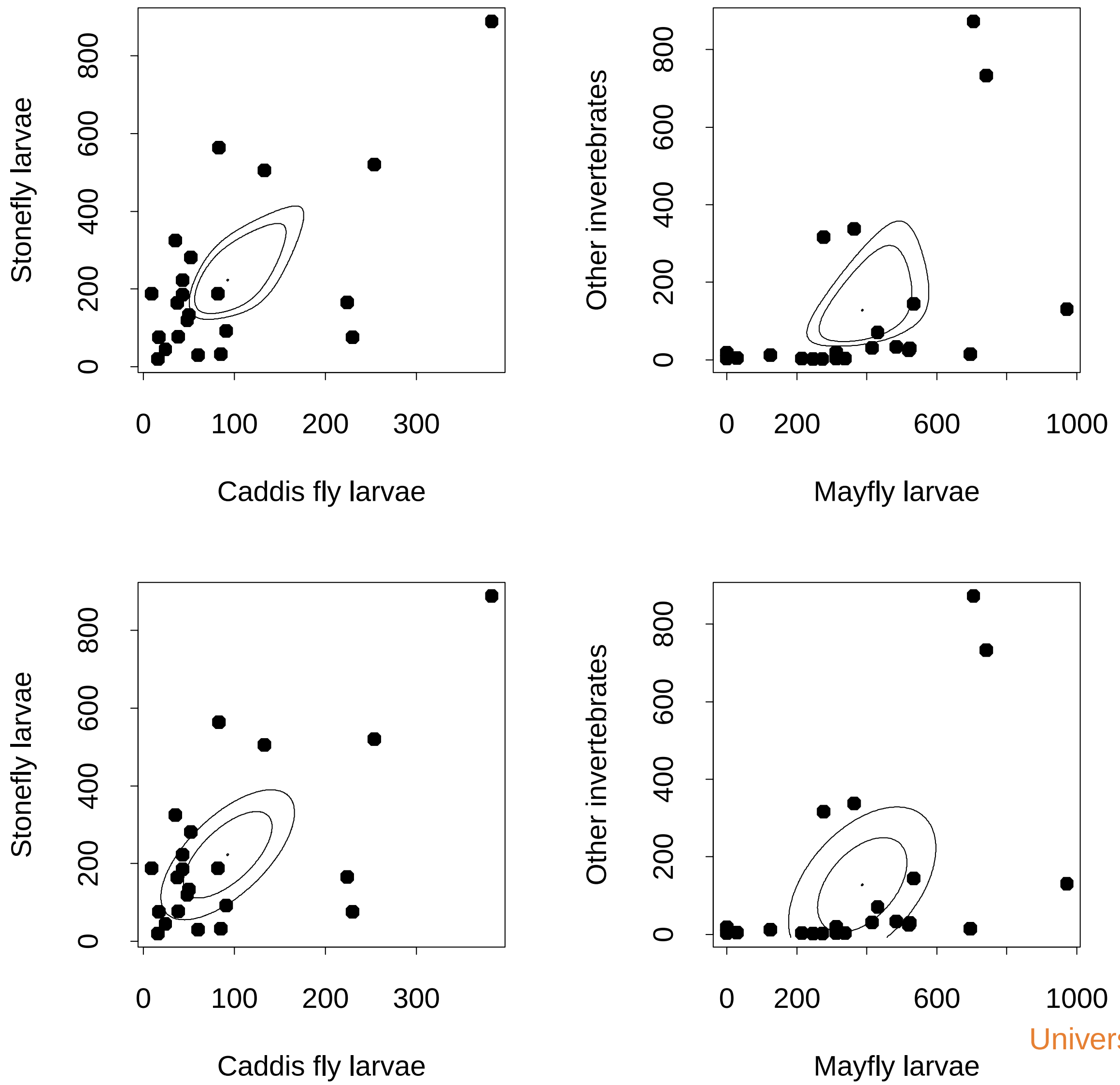
Coverage error $O(n^{-2})$

Same error rate as bootstrapping the bootstrap

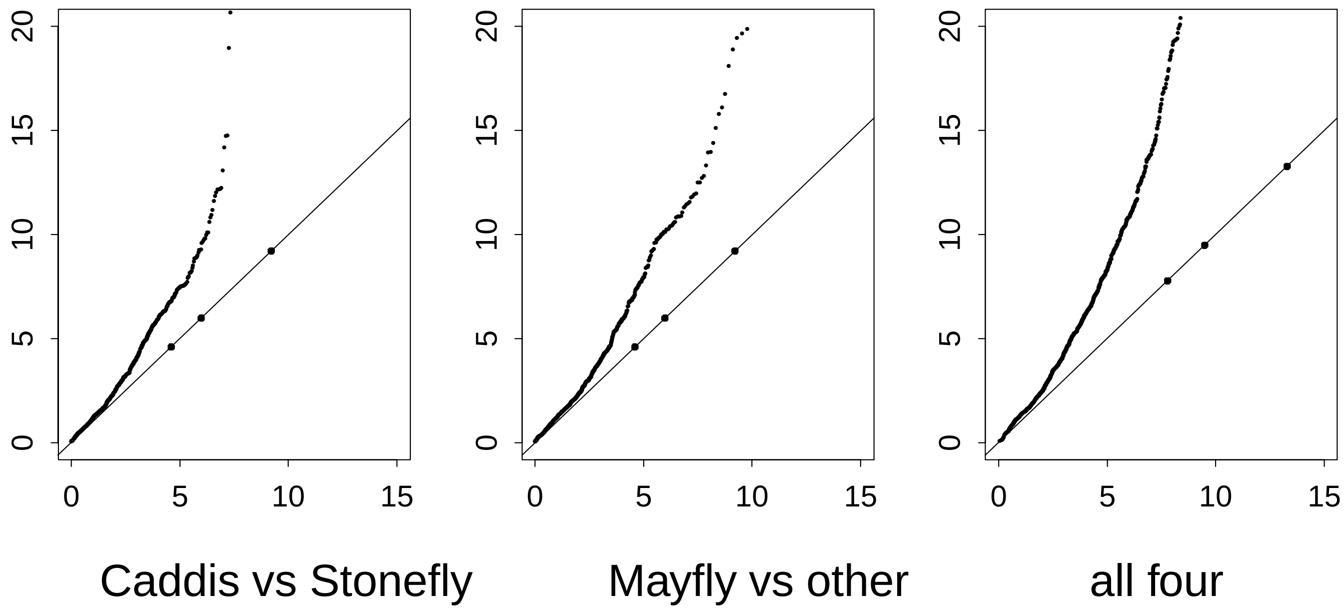
Sets in faster than Bartlett correction

Need further adjustments for one-sided inference

Bootstrap (and χ^2) calibrated Dipper regions



Resampled $-2\log\mathcal{R}(\mu)$ values vs χ^2



Smooth functions of means

$$\sigma = \sqrt{E(X^2) - E(X)^2}$$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E(X)^2}\sqrt{E(Y^2) - E(Y)^2}}$$

$$\theta = h(E(U, V, \ldots, Z)$$

Generally

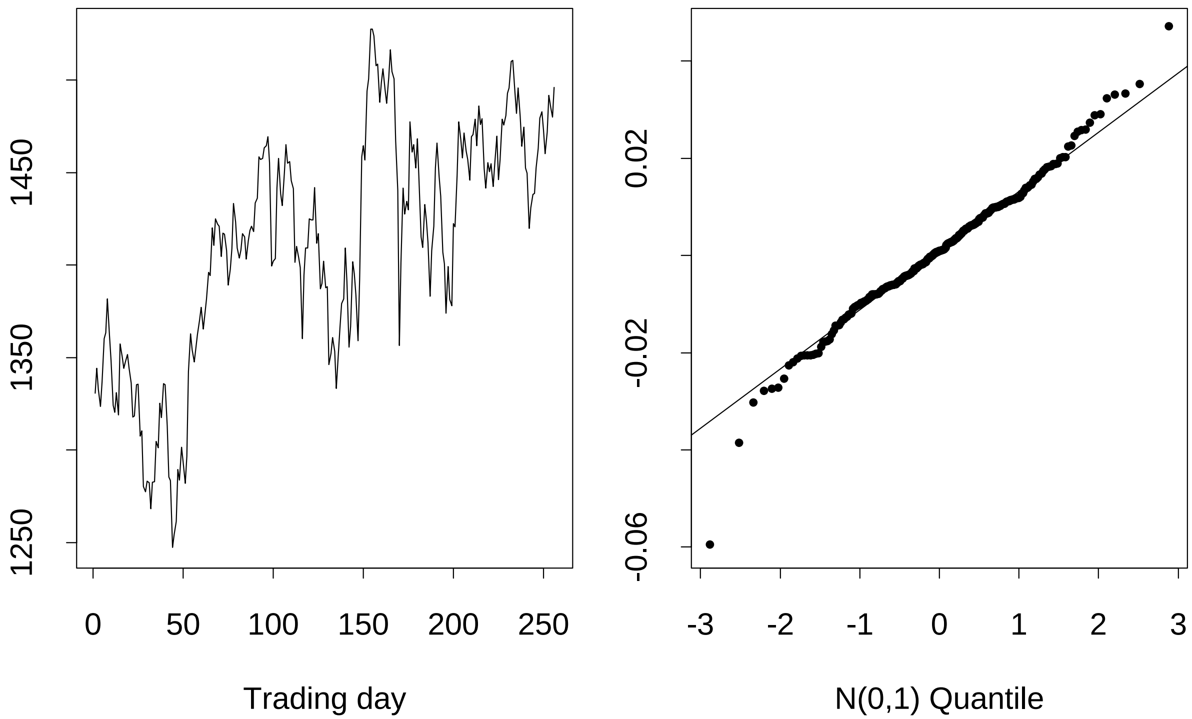
$$X = (U, V, \ldots, Z)$$

$$\theta = E(h(X))$$

$$\hat{\theta} = h(\bar{x}) \doteq h(E(X)) + (\bar{x} - E(X))' \frac{\partial}{\partial x} h(E(X))$$

h nearly linear near $E(X) \implies \theta$ nearly a mean

S&P 500 returns

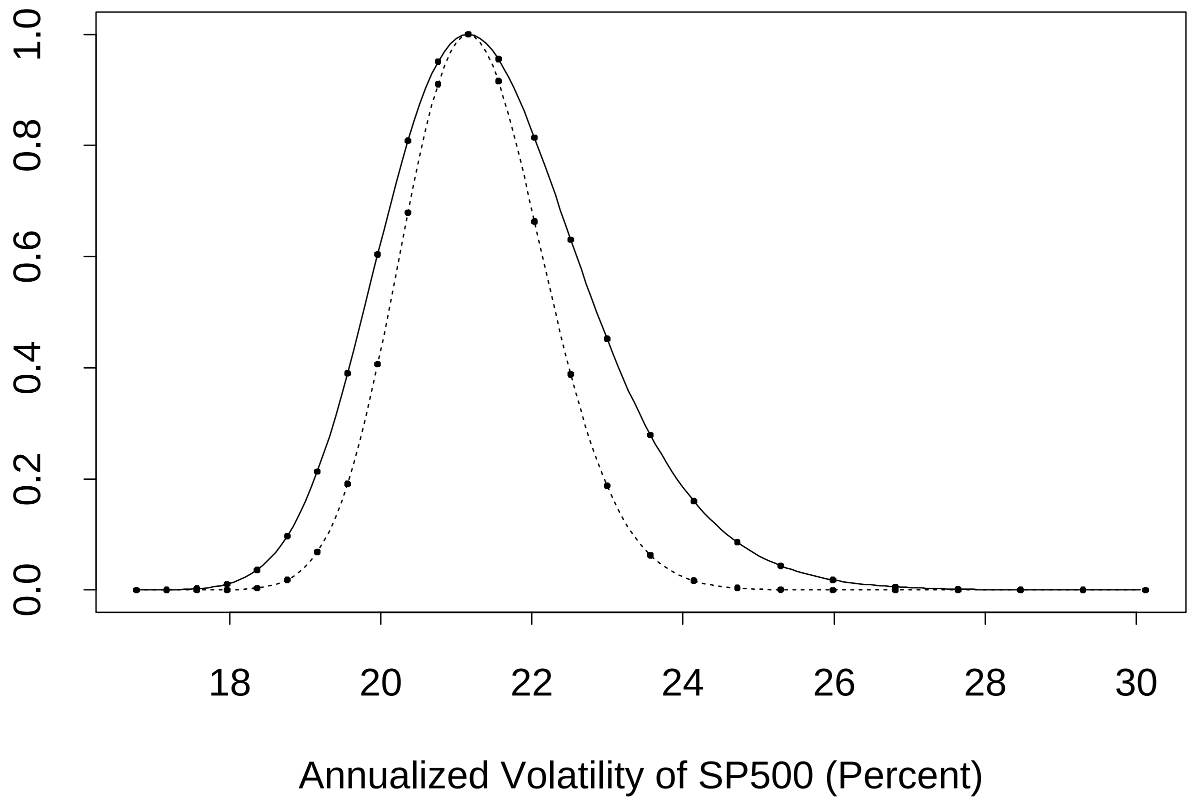


Return $= \log(x_{i+1}/x_i)$

Nearly $N(0, \sigma^2)$ but heavy tails

Volatility σ is Standard deviation of returns

S&P 500 returns



Solid = Empirical likelihood

Dashed = Normal likelihood

Estimating equations

More powerful and general than smooth functions

Define θ via $E(m(X,\theta)) = 0$

Define $\hat{\theta}$ via $\frac{1}{n}\sum_{i=1}^{n} m(x_i,\hat{\theta}) = 0$

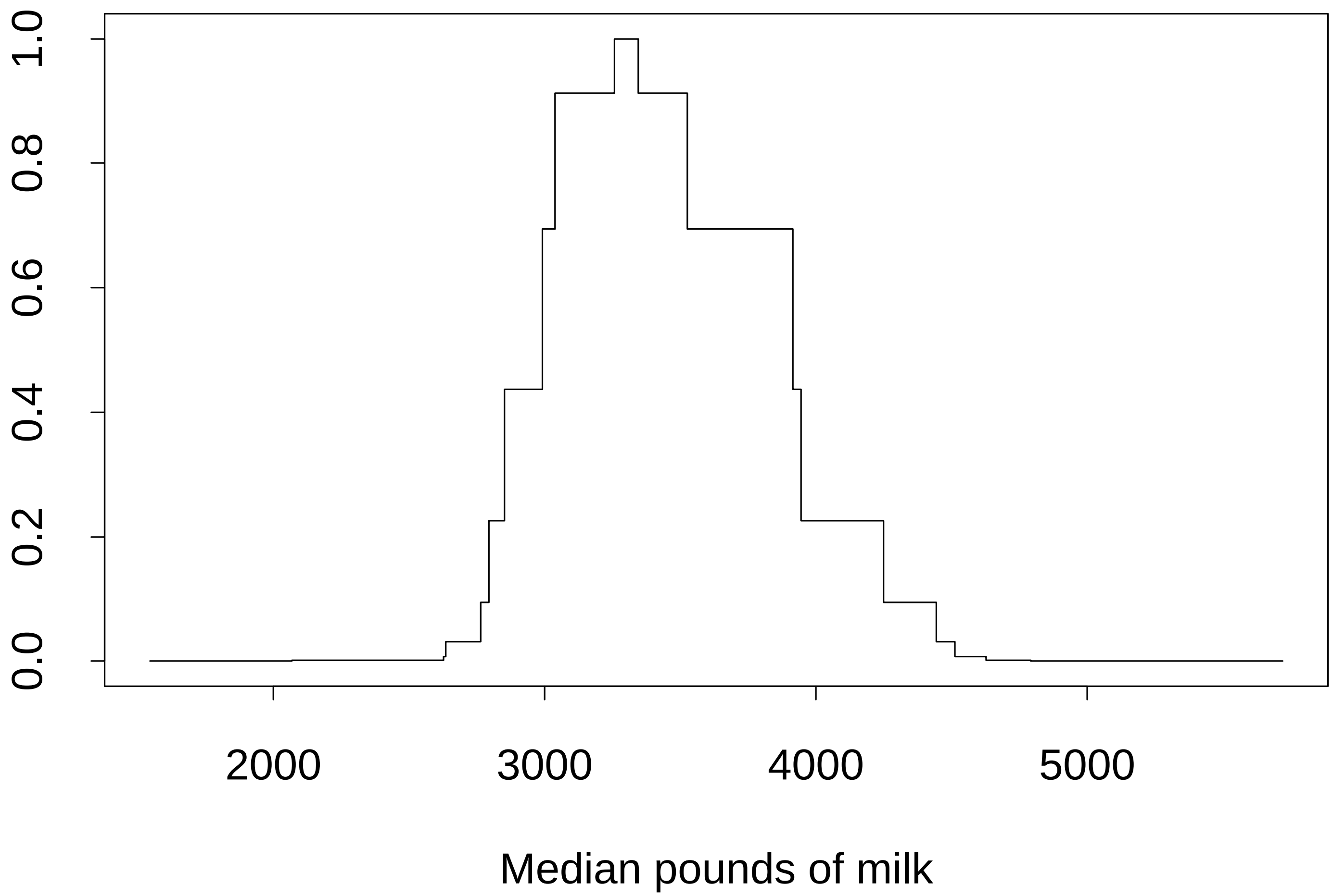
Usually $\dim(m) = \dim(\theta)$

Basic examples: $\dim(m) = \dim(\theta) = 1$

$m(X,\theta)$	Statistic
$X-\theta$	Mean
$1_{X\in A}-\theta$	Probability of set A
$1_{X\le\theta}-\frac{1}{2}$	Median
$\frac{\partial}{\partial x}\log(f(X;\theta))$	MLE under f

$$-2\log\mathcal{R}(\theta_0) \to \chi^2_{\text{Rank}(Var(m(X,\theta_0)))}$$

Empirical likelihood for a median



LR is constant between observations

Est. eq. with nuisance parameters

For $\theta = (\rho)$ and $\nu = (\mu_x, \mu_y, \sigma_x, \sigma_y)$
$E(m(X, \theta, \nu)) = 0 = \frac{1}{n} \sum_{i=1}^{n} m(X_i, \hat{\theta}, \hat{\nu})$

Correlation example

$$0 = E(X - \mu_x)$$
$$0 = E(Y - \mu_y)$$
$$0 = E((X - \mu_x)^2 - \sigma_x^2)$$
$$0 = E((Y - \mu_y)^2 - \sigma_y^2)$$
$$0 = E((X - \mu_x)(Y - \mu_y) - \rho\sigma_x\sigma_y)$$

Profile empirical likelihood $\mathcal{R}(\theta) = \sup_{\nu} \mathcal{R}(\theta, \nu)$
Typically $-2 \log \mathcal{R}(\theta_0) \to \chi^2_{\dim(\theta)}$

Huber's robust estimation

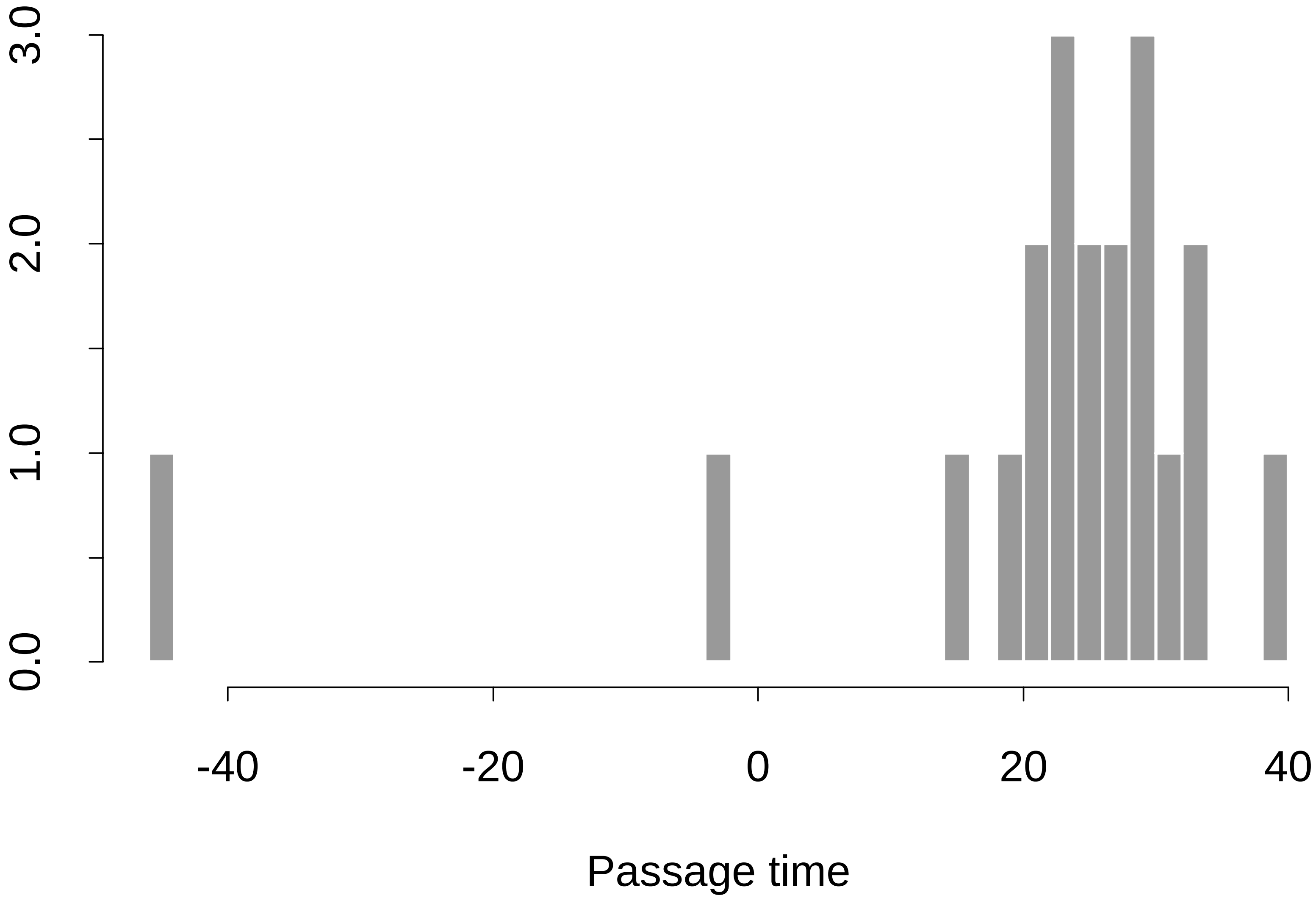
$$0 = \frac{1}{n}\sum_{i=1}^{n}\psi\Big(\frac{x_i-\mu}{\sigma}\Big)0 = \frac{1}{n}\sum_{i=1}^{n}\Big[\psi\Big(\frac{x_i-\mu}{\sigma}\Big)^2 - 1\Big]$$

Like mean for small obs, median for outliers

$$\psi(z) = \begin{cases} z, & |z| \le 1.35 \\ 1.35\,\mathrm{sign}(z), & |z| \ge 1.35. \end{cases}$$

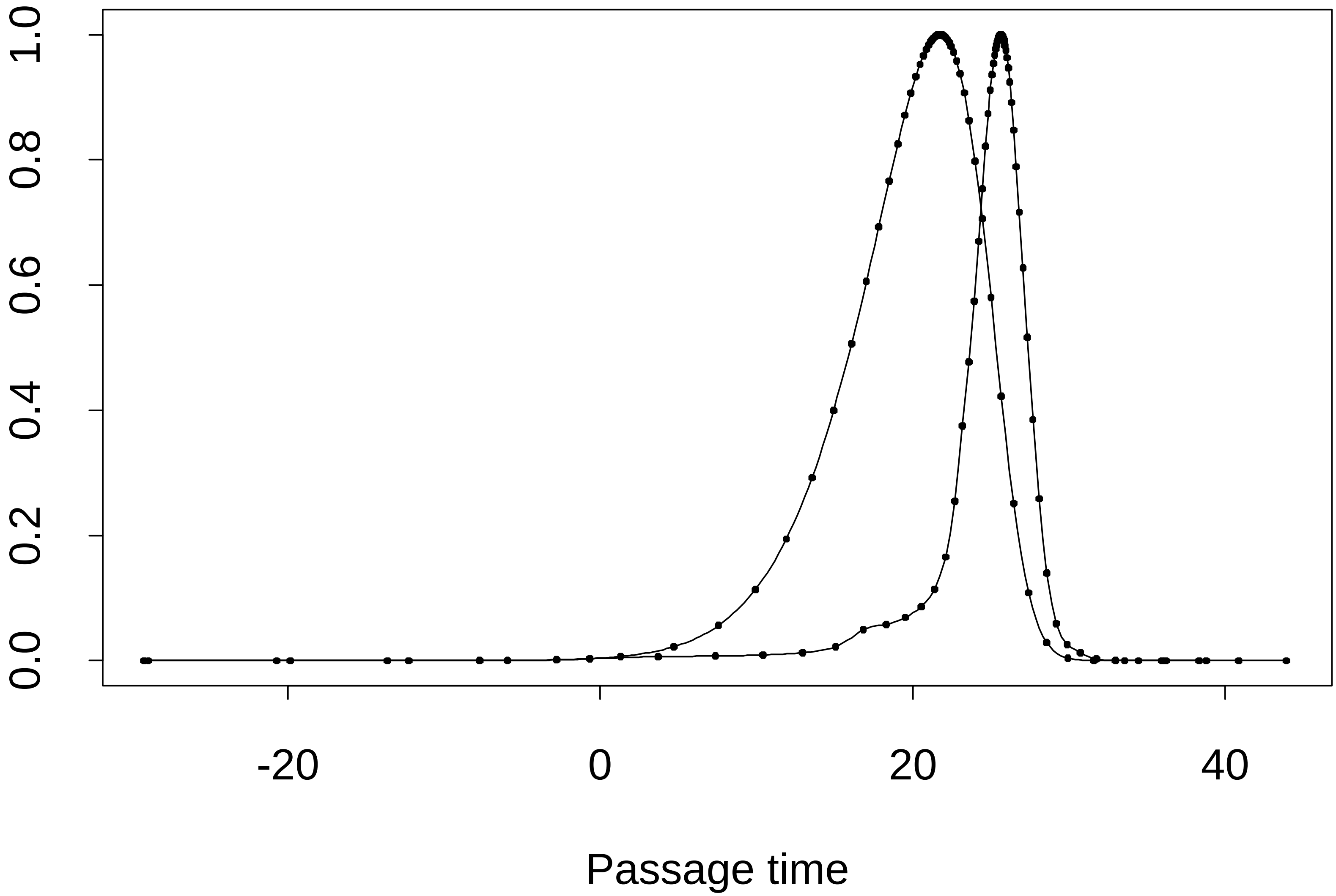
$$\mathcal{R}(\mu) = \max_{\sigma}\max\Big\{\prod_{i=1}^{n} nw_i \mid 0 \le w_i, \sum_i w_i = 1, \sum_i w_i\psi\Big(\frac{x_i-\mu}{\sigma}\Big) = 0,$$
$$\sum_i w_i\Big[\psi\Big(\frac{x_i-\mu}{\sigma}\Big)^2 - 1\Big] = 0\Big\}$$

Newcomb’s passage times of light



From Stigler

EL for mean and Huber's location



Maximum empirical likelihood estimates

Hartley & Rao	1968	means & finite populations
Owen	1991	means IID
Qin & Lawless	1993	estimating eqns IID

Simple MELEs

Observe (X_i, Y_i) pairs with mean (μ_x, μ_y) and $\mu_x = \mu_{x0}$ *known*
Let w_i maximize $\prod_{i=1}^n n w_i$ st:
$w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$ and $\sum_{i=1}^n w_i x_i = \mu_x$

$$\text{MELE} \quad \widetilde{\mu}_y = \sum_{i=1}^n w_i y_i \doteq \bar{Y} - \Sigma_{yx} \Sigma_{xx}^{-1} (\bar{X} - \mu_{x0})$$

Conditional empirical likelihood

$\mu_x = \mu_{x0}$ known

$$\mathcal{R}_{X,Y}(\mu_x, \mu_y) = \max\Big\{\prod_{i=1}^n nw_i \mid w_i \geq 0, \sum_i w_i x_i = \mu_x, \sum_i w_i y_i = \mu_y\Big\}$$

$$\mathcal{R}_X(\mu_x) = \max\Big\{\prod_{i=1}^n nw_i \mid w_i \geq 0, \sum_i w_i x_i = \mu_x\Big\}$$

$$\mathcal{R}_{Y|X}(\mu_y \mid \mu_x) = \frac{\mathcal{R}_{X,Y}(\mu_x, \mu_y)}{\mathcal{R}_X(\mu_x)}$$

$$-2\log \mathcal{R}_{Y|X}(\mu_y \mid \mu_{x0}) \to \chi^2_{\dim(Y)}$$

$$-2\log \mathcal{R}_Y \doteq n(\mu_{y0} - \bar{y})'\Sigma_{yy}^{-1}(\mu_{y0} - \widetilde{\mu}_y)$$

$$-2\log \mathcal{R}_{Y|X} \doteq n(\mu_{y0} - \widetilde{\mu}_y)'\Sigma_{y|x}^{-1}(\mu_{y0} - \widetilde{\mu}_y)$$

$$\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy} \leq \Sigma_{yy}$$

Side or auxiliary information

Known parameter	Estimating equation
mean	$X - \mu_x$
α quantile	$1_{X \leq Q} - \alpha$
$P(A \mid B)$	$(1_A - \rho)1_B$
$E(X \mid B)$	$(X - \mu)1_B$

Overdetermined equations

$$E(m(X,\theta)) = 0, \quad \dim(m) > \dim(\theta)$$

Approaches:

1. Drop $\dim(m) - \dim(\theta)$ equations
2. Replace $m(X,\theta)$ by $m(X,\theta)A(\theta)$ where
   A a $\dim(m) \times \dim(\theta)$ matrix (IE pick $\dim(\theta)$ linear comb. of m)
3. GMM: estimate the optimal A
4. MELE: $\widetilde{\theta} = \arg\max_\theta \max_{w_i} \prod_i nw_i$ st $\sum_{i=1}^n w_i m(x_i,\theta) = 0$

MELE has same asymptotic variance as using optimal $A(\theta)$

Bias scales more favorably with dimensions for MELE than for $\hat{A}$ methods

Qin and Lawless result

$\dim(m) = p + q \geq p = \dim(\theta)$ MELE $\widetilde{\theta}$

$$-2\log(\mathcal{R}(\theta_0)/\mathcal{R}(\widetilde{\theta})) \to \chi^2_{(p)}$$ conf regions for θ_0

$$-2\log\mathcal{R}(\widetilde{\theta}) \to \chi^2_{(q)}$$ goodness of fit tests when $q > 0$

Requires considerable smoothness

What happens for IQR $= Q^{0.75} - Q^{0.25}$?

$$0 = E(1_{X \leq Q^{.75}} - 0.75) = E(1_{X \leq Q^{.25}} - 0.25)$$

$$0 = E(1_{X \leq Q^{.25}+IQR} - 0.75) = E(1_{X \leq Q^{.25}} - 0.25)$$

Need to max over $Q^{.25}$

Euclidean log likelihood

Replace $-\sum_{i=1}^n \log(nw_i)$ by

$$\ell_E = -\frac{1}{2}\sum_{i=1}^n (nw_i - 1)^2$$

Reduces to Hotelling's T^2 for the mean Owen

Reduces to Huber-White covariance for regression

Reduces to continuous updating GMM Kitamura

Quadratic approx to EL, like Wald test is to parametric likelihood

Allows $w_i < 0$, and so

1. confidence regions for means can get out of the convex hull
2. confidence regions no longer obey range restrictions

Exponential empirical likelihood

Replace $-\sum_{i=1}^{n} \log(nw_i)$ by

$$\text{KL} = \sum_{i=1}^{n} w_i \log(nw_i)$$

relates to entropy and exponential tilting

Hellinger distance

$$\sum_{i=1}^{n} (w_i^{1/2} - n^{-1/2})^2$$

Renyi, Cressie-Read

$$\frac{2}{\lambda(\lambda+1)}\sum_{i=1}^{n}((nw_i)^{-\lambda}-1)$$

λ	Method
-2	Euclidean log likelihood
$\to -1$	Exponential empirical likelihood
$-1/2$	Freeman-Tukey
$\to 0$	Empirical likelihood
1	Pearson's

Alternate artificial likelihoods

All Renyi Cressie-Read familiies have χ^2 calibrations. Baggerly

Only EL is Bartlett correctable Baggerly

$-2\sum_{i=1}^n \widetilde{\log}(nw_i)$ Bartlett correctable if

$$\widetilde{\log}(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + o(z^4), \quad \text{as } z \to 0$$

Corcoran

Regression

$$E(Y \mid X = x) \doteq \beta_0 + \beta_1 x$$

Models (Freedman)

Correlation $(X_i, Y_i) \sim F_{XY}$ IID

Regression x_i fixed, $Y_i \sim F_{Y|X=(1,x_i)}$ indep

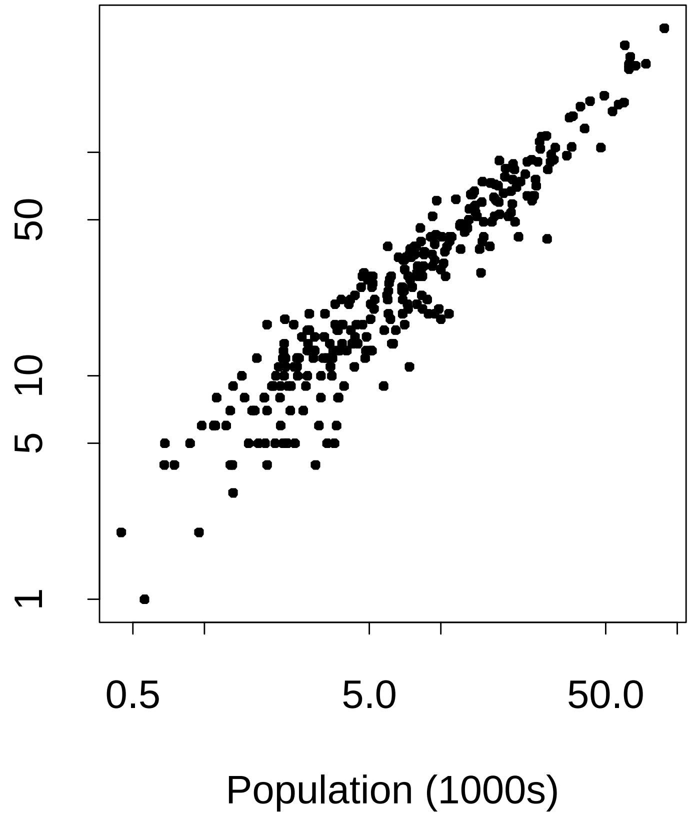
Correlation model

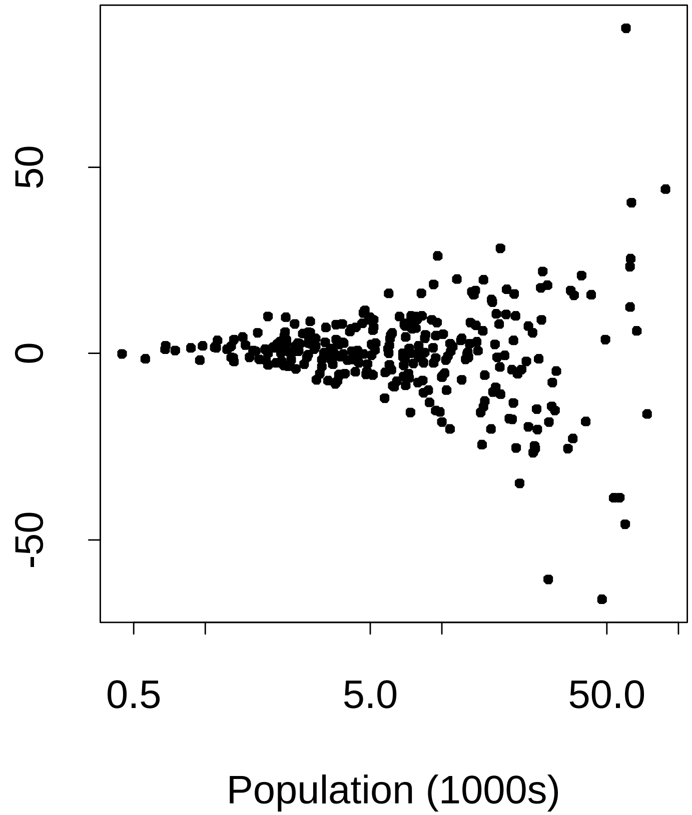
$$\beta = E(X'X)^{-1} E(X'Y)$$

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^{n} X_i' X_i \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} X_i' Y_i$$

β and $\hat{\beta}$ well defined even for lack of fit

Cancer deaths vs population, by county





Nearly linear regression nonconstant residual variance

Royall via Rice

Estimating equations for regression

$$E(X'(Y - X'\beta)) = 0, \qquad \frac{1}{n}\sum_{i=1}^{n} X_i'(Y_i - X_i'\hat{\beta}) = 0$$

$$\mathcal{R}(\beta) = \max\Big\{\prod_{i=1}^{n} nw_i \mid \sum_{i=1}^{n} w_i Z_i(\beta) = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1\Big\}$$

$$Z_i(\beta) = X_i'(Y_i - X_i'\beta)$$

need $E(\|Z\|^2) \leq E\Big(\|X\|^2(Y - X'\beta)^2\Big) < \infty$

Don't need:

normality, constant variance, exact linearity

For cancer data

$P_i =$ population of i'th county in 1000s

$C_i =$ cancer deaths of i'th county in 20 years

$C_i \doteq \beta_0 + \beta_1 P_i$

$\hat{\beta}_1 = 3.58 \qquad \Longrightarrow \; 3.58/20 = 0.18$ deaths per thousand per year

$\hat{\beta}_0 = -0.53$ near zero, as we'd expect

Regression through the origin

$$C_i \doteq \beta_1 P_i$$

Residuals should have mean zero and be orthogonal to P_i

We want two equations in one unknown β_1

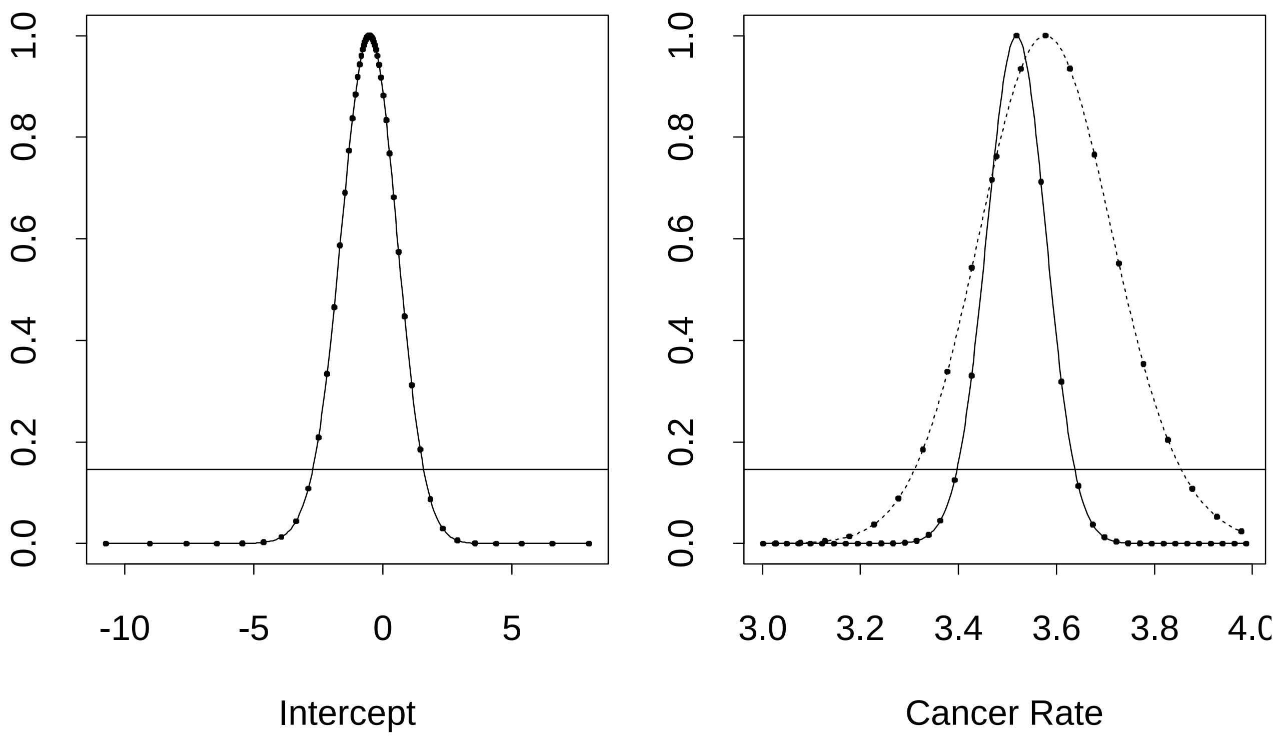
Equivalently, side information $\beta_0 = 0$

Least squares regression through origin does not solve both equations

MELE $\widetilde{\beta}_1 = \arg\max_{\beta_1} \mathcal{R}(\beta_1)$

$$\mathcal{R}(\beta_1) = \max\Big\{\prod_{i=1}^n nw_i \mid \sum_{i=1}^n w_i(C_i - P_i\beta_1) = 0, \\ \sum_{i=1}^n w_iP_i(C_i - P_i\beta_1) = 0, \sum_{i=1}^n w_i = 1, w_i \geq 0\Big\}$$

Regression parameters



Intercept nearly 0, MELE smaller than MLE

CI based on conditional empirical likelihood

Constraint narrows CI for slope by over half

Fixed predictor regression model

$E(Y_i) = \mu_i \doteq \beta_0 + \beta_1 x_i$ fixed, and $V(Y_i) = \sigma_i^2$

With lack of fit $\mu_i \neq \beta_0 + \beta_1 x_i$

No good definition of 'true' β given L.O.F.

$Z_i = x_i(Y_i - x_i'\beta)$ have

1. mean $E(Z_i) = x_i(\mu_i - x_i'\beta)$ $\quad$ 0 may be the common value
2. variance $V(Z_i) = x_i x_i' \sigma_i^2$ $\quad$ non-constant, even if σ_i^2 constant

Triangular array ELT

$$
\begin{array}{cccccc}
Z_{11} & & & & & \\
Z_{12} & Z_{22} & & & & \\
Z_{13} & Z_{23} & Z_{33} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
Z_{1n} & Z_{2n} & Z_{3n} & \cdots & Z_{nn} & \\
\vdots & \vdots & \vdots & & & \ddots
\end{array}
$$

Row n has indep $Z_{1n}, \dots, Z_{nn}$, common mean 0 not ident distributed

Different rows have different distns

Still get $-\log \mathcal{R}($Common mean $= 0) \to \chi^2_{\dim(Z)}$ under mild conditions

Applies for fixed x regression: $Z_{in} = x_i(Y_i - x_i'\beta)$

Variance modelling

Working model $Y \sim N(x'\beta, e^{2z'\gamma})$

$$0 = \frac{1}{n}\sum_{i=1}^{n} x_i(y_i - x_i'\beta)\, e^{-2z_i'\gamma} \qquad \text{(weight} \propto 1/\text{var)}$$

$$0 = \frac{1}{n}\sum_{i=1}^{n} z_i\Big(1 - \exp(-2z_i'\gamma)(y_i - x_i'\beta)^2\Big)$$

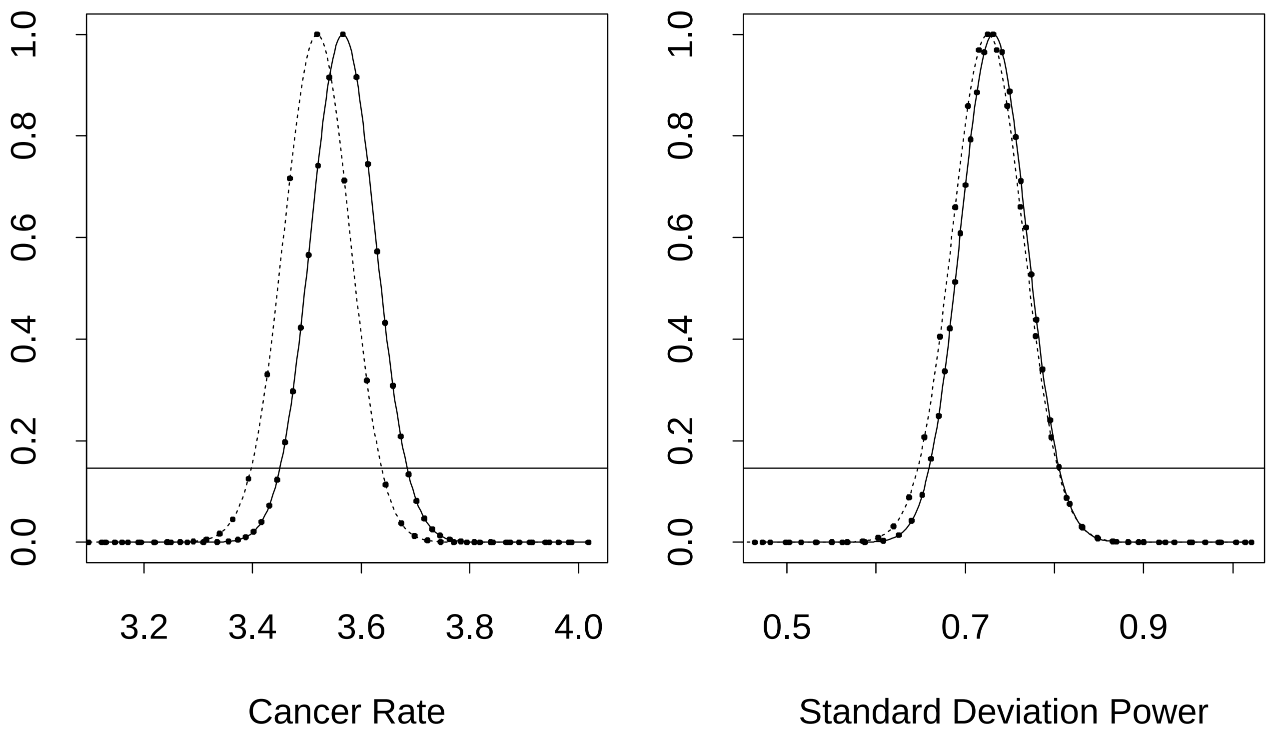
For cancer data

$$x_i = (1, P_i) \quad z_i = (1, \log(P_i))$$

$$E(Y_i) = \beta_0 + \beta_1 P_i \quad \sqrt{V(Y_i)} = \exp(\gamma_0 + \gamma_1 \log(P_i)) = e^{\gamma_0} P_i^{\gamma_1}$$

and $\beta_0 = 0$

Heteroscedastic model

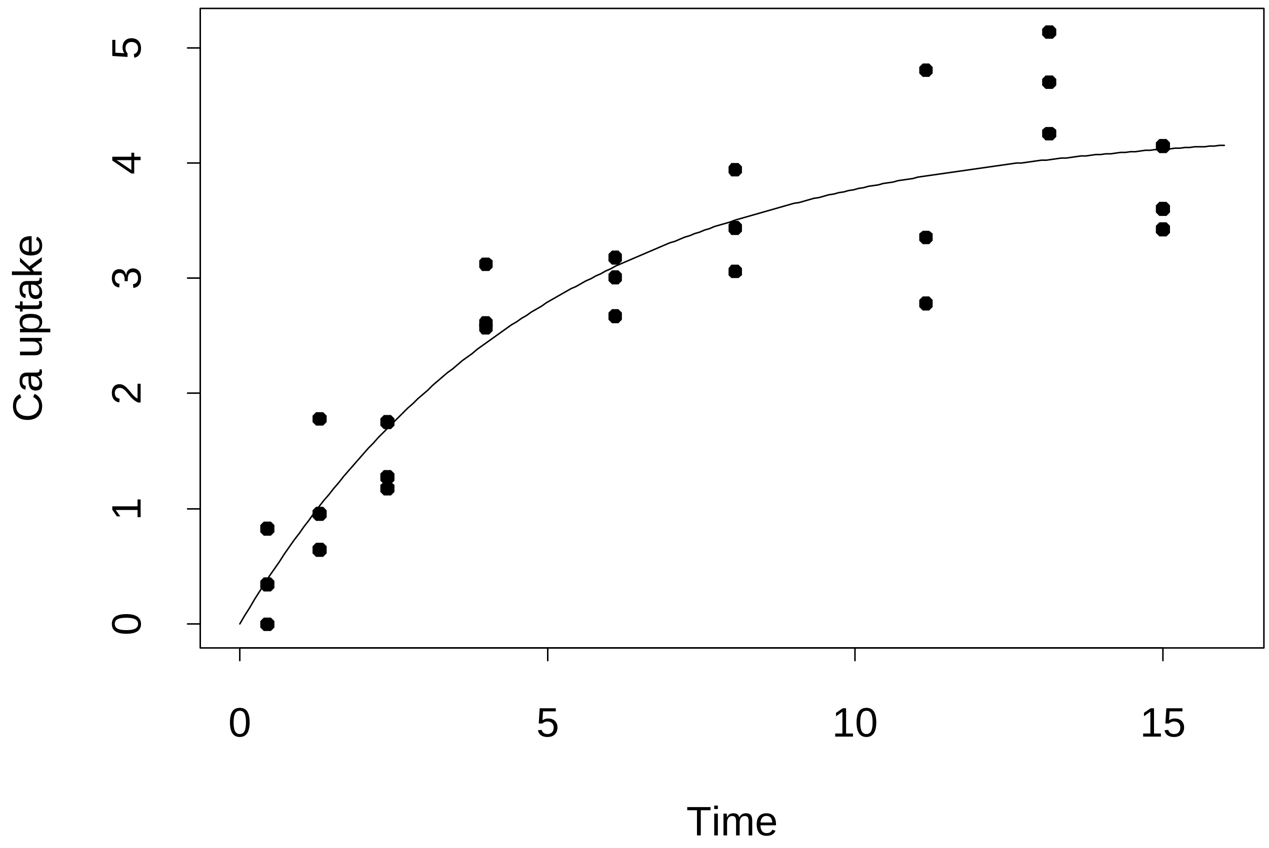


Left: solid curve accounts for nonconstant variance

Right: solid curve forces $\beta_0 = 0$, and,

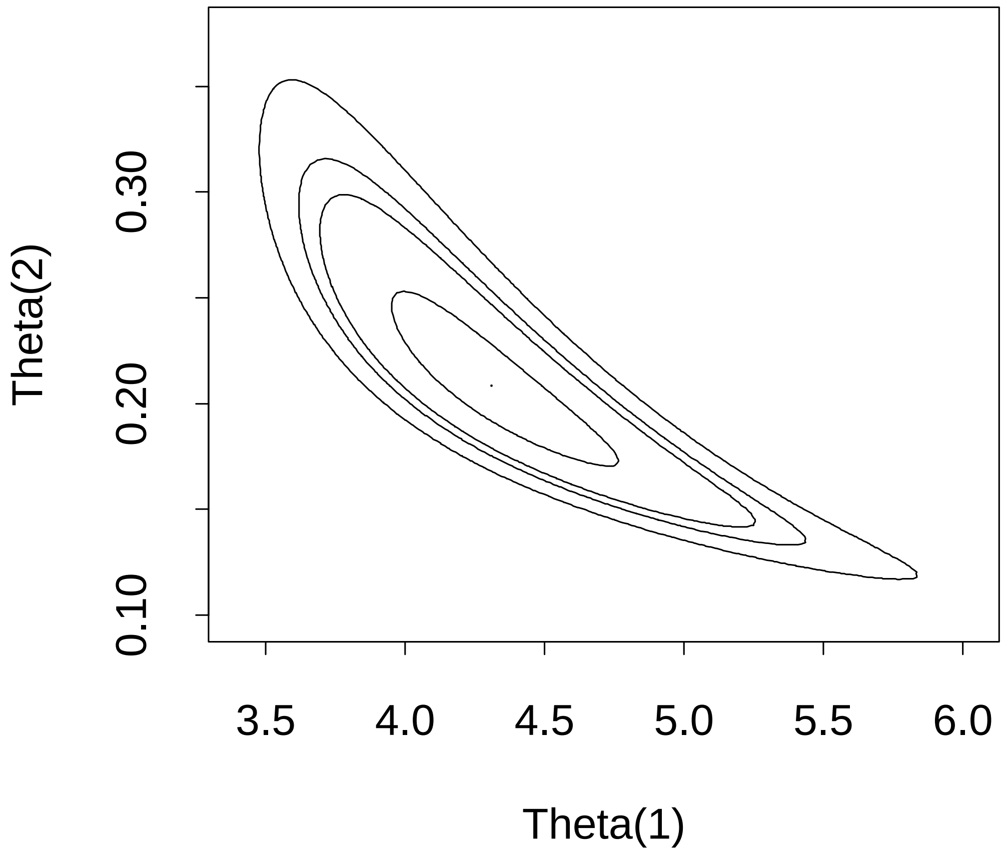
rules out $\gamma_1 = 1/2$ (Poisson) and $\gamma_1 = 1$ (Gamma)

Nonlinear regression



$$y \doteq f(x, \theta) \equiv \theta_1(1 - \exp(-\theta_2 x))$$

Nonlinear regression regions



$$0 = \sum_{i=1}^{n} w_i (Y_i - f(x_i, \theta)) \frac{\partial}{\partial \theta} f(x_i, \theta)$$

Don't need: normality or constant variance

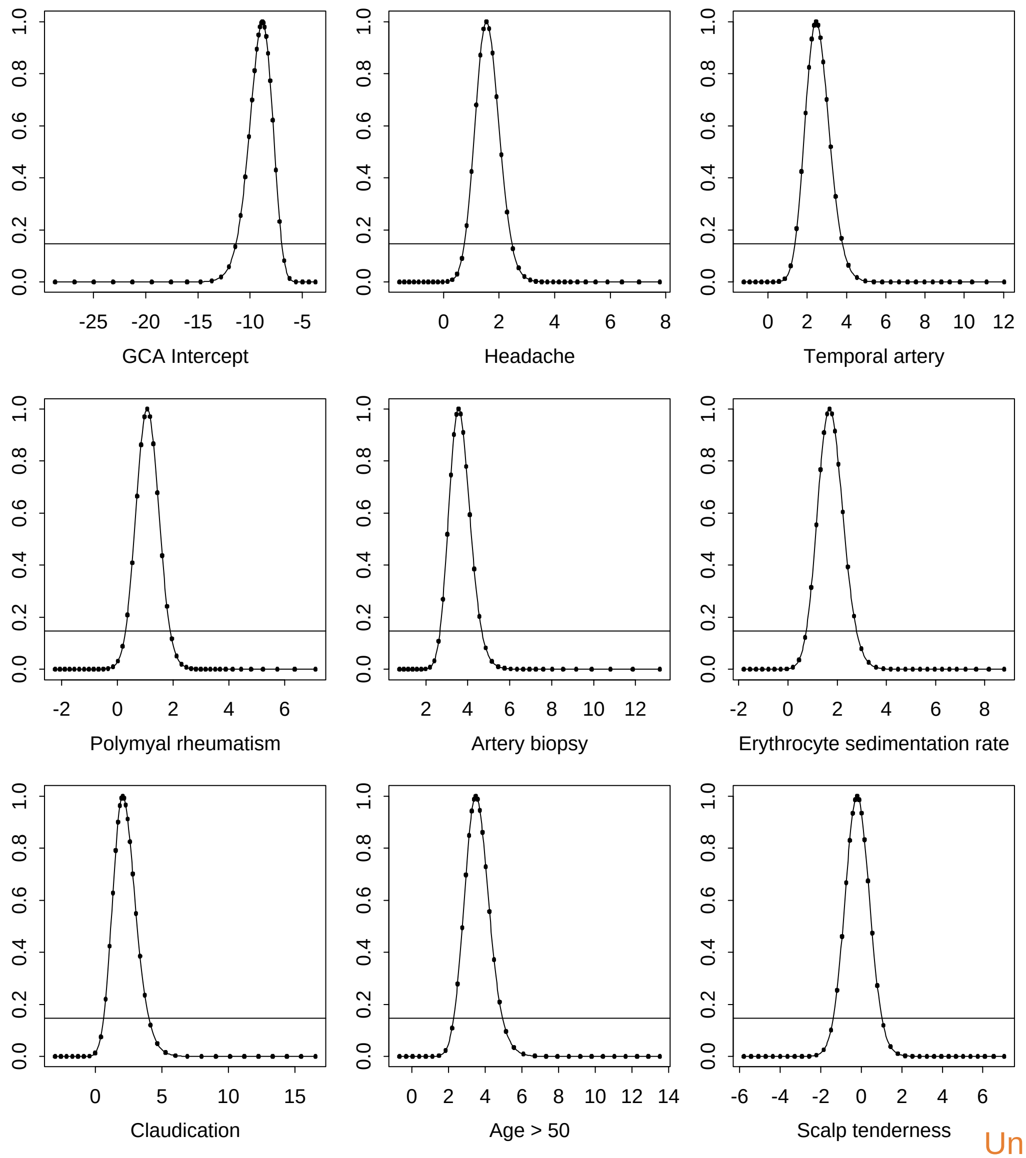
Logistic regression

- Giant cell arteritis is a type of vasculitis (inflamation of blood or lymph vessels)
- Not all vasculitis is GCA
- Try to predict GCA from 8 binary predictors

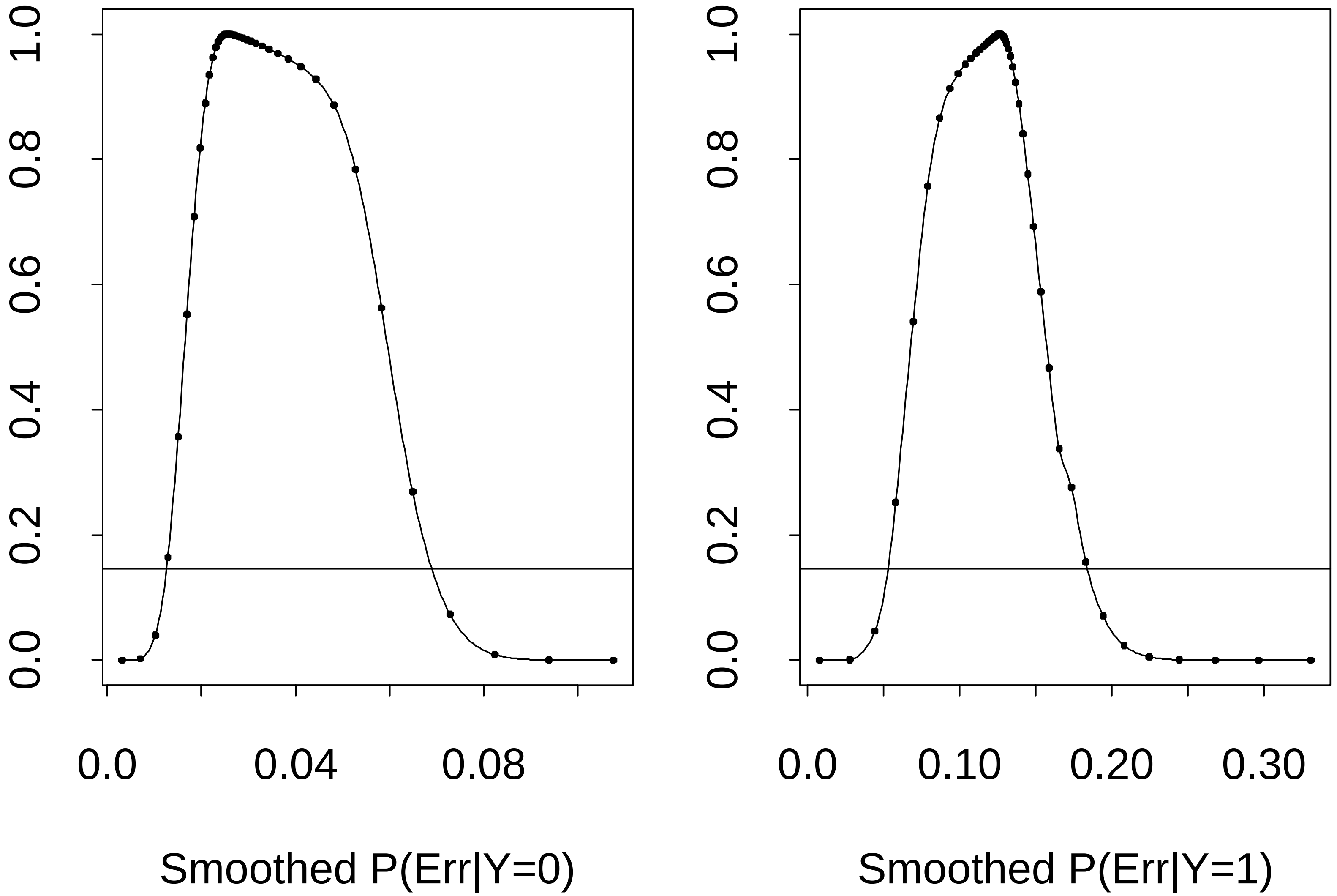
$$\Pr(GCA) \doteq \tau(X'\beta) = \frac{\exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_8 X_8)}{1 + \exp(\beta_0 + \beta_1 X_1 + \cdots + \beta_8 X_8)}$$

Likelihood estimating equations reduce to: $Z_i(\beta) = X_i(Y_i - \tau(X'\beta))$

Logistic regression coefficients



Prediction accuracy



Biased sampling

Examples

1. Sample children, then record family sizes.
2. Draw blue line over cotton, sample fibers that are partly blue.
3. When $Y = y$ it is recorded as X with prob. $u(y)$, lost with prob. $1 - u(y)$.

$Y \sim F$, observe $X \sim G$, but we really want F

$$G(A) = \frac{\int_A u(y)\, dF(y)}{\int u(y)\, dF(y)}$$

$$L(F) = \prod_{i=1}^{n} G(\{x_i\}) = \prod_{i=1}^{n} \frac{F(\{x_i\})\, u(x_i)}{\int u(x)\, dF(x)}$$

NPMLE

$$\widehat{G}(\{x_i\}) = \frac{1}{n} \qquad \text{(for simplicity, suppose no ties)}$$

$$\widehat{G}(\{x_i\}) \propto \hat{F}(\{x_i\}) \times u(x_i)$$

$$\widehat{F}(\{x_i\}) = \frac{u_i^{-1}}{\sum_{j=1}^n u_j^{-1}}$$

For the mean

$$\hat{\mu} = \frac{\sum_{i=1}^n x_i/u_i}{\sum_{i=1}^n 1/u_i}$$

Horvitz-Thompson estimator is NPMLE

$$\hat{\mu} = \left(\frac{1}{n}\sum_{i=1}^n x_i^{-1}\right)^{-1}$$

when $u_i \propto x_i$, so length bias $\Longrightarrow$ harmonic mean

Biased sampling again

$$0 = \int m(x,\theta)dF(x) = \int \frac{m(x,\theta)}{u(x)} dG(x)$$

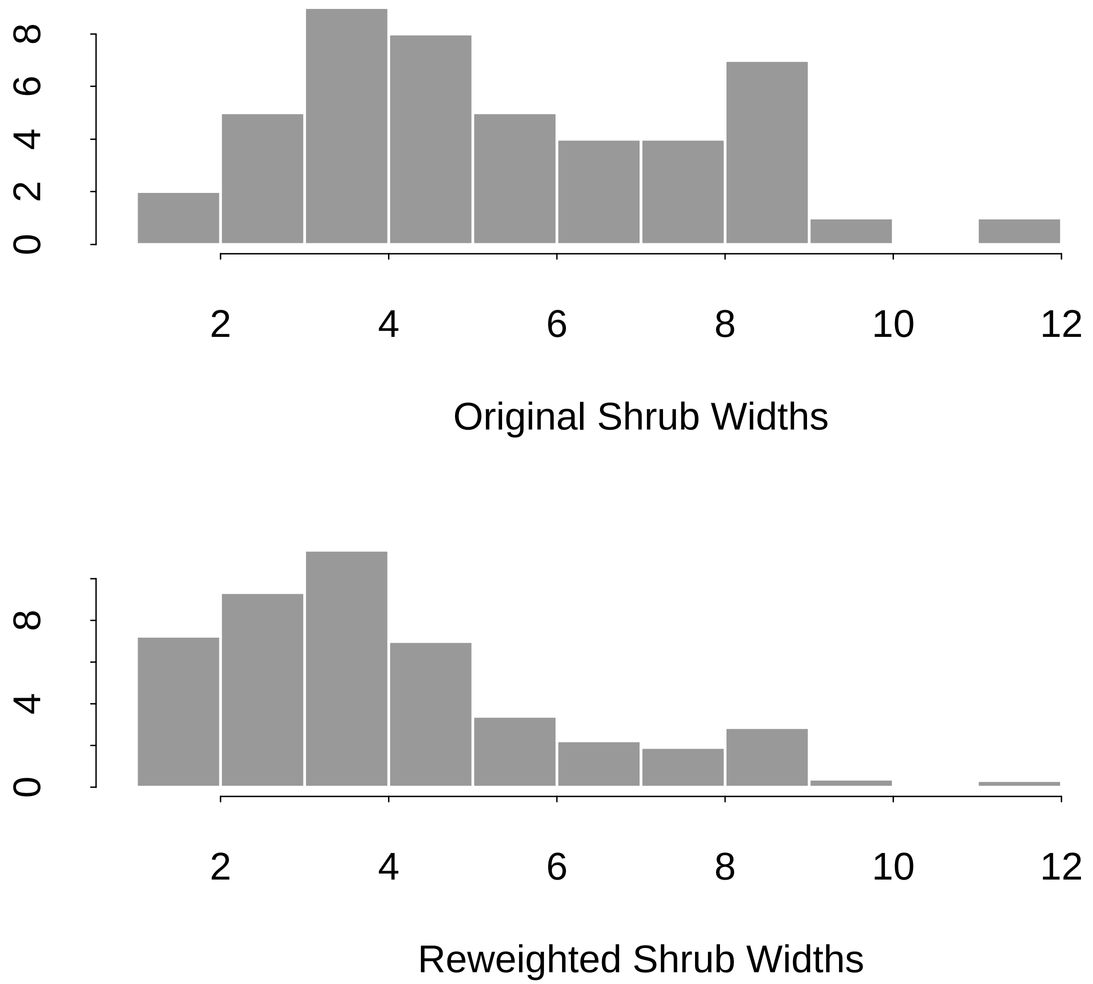
$$G(\{x_i\}) = w_i \implies F(\{x_i\}) = \frac{w_i/u_i}{\sum_{j=1}^n 1/u_j}$$

Very simple recipe

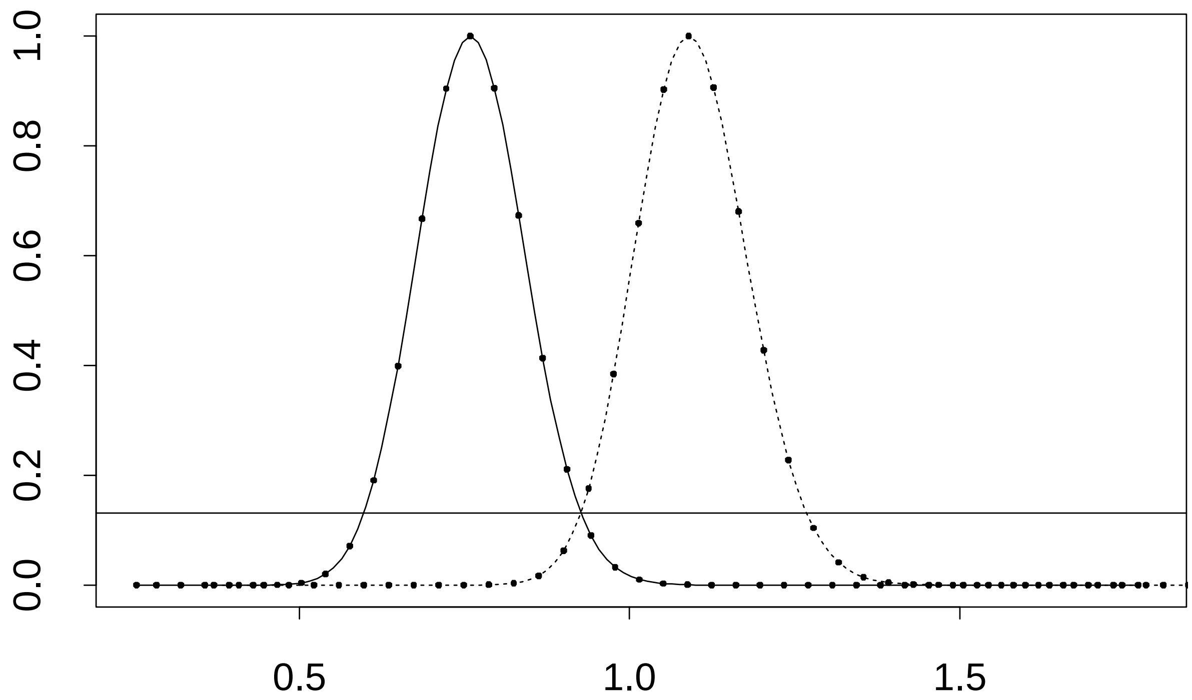
$$m(x,\theta) \quad \longrightarrow \quad \widetilde{m}(x,\theta) \equiv \frac{m(x,\theta)}{u(x)}$$

$$\mathcal{R}(\theta) = \max\left\{ \prod_{i=1}^n nw_i \mid w_i \geq 0, \sum_{i=1}^n w_i = 1, \sum_{i=1}^n w_i \, \widetilde{m}(x_i,\theta) = 0 \right\}$$

Transect sampling of shrubs (Muttlak & McDonald)



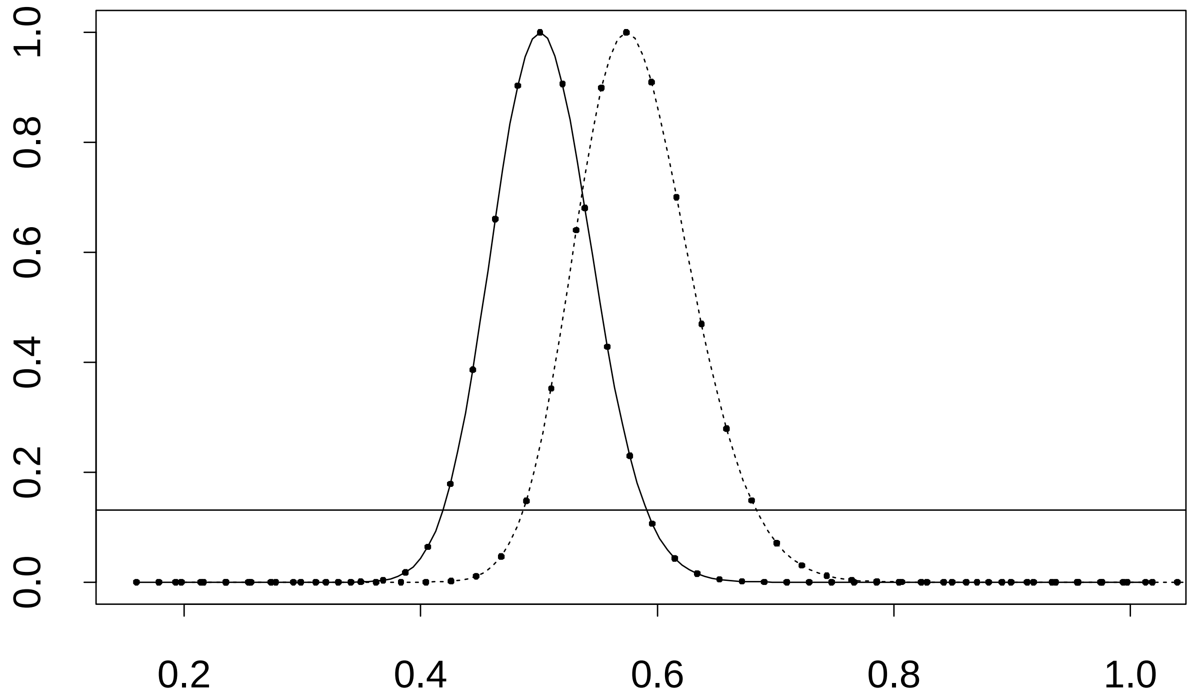
Mean shrub width



$$0 = \sum_{i=1}^{n} w_i \frac{x_i - \mu}{x_i}$$ Solid

$$0 = \sum_{i=1}^{n} w_i (x_i - \mu)$$ Dotted

Standard dev. of shrub width



$$0 = \sum_{i=1}^{n} w_i \frac{(x_i - \mu)^2 - \sigma^2}{x_i}$$ Solid

$$0 = \sum_{i=1}^{n} w_i((x_i - \mu)^2 - \sigma^2)$$ Dotted

Multiple biased samples

Population k sampled from F with bias $u_k(\cdot)$, $k = 1, \ldots, s$

$$X_{ik} \sim G_k, \qquad i = 1, \ldots, n_k, \quad k = 1, \ldots, s$$

$$G_k(A) = \frac{\int_A u_k(y)\, dF(y)}{\int u_k(y)\, dF(y)}, \quad k = 1, \ldots, s$$

Examples

1. clinical trials with varying enrolment criteria
2. mix of length biased and unbiased samples
3. telescopes with varying detection limits
4. sampling from different frames

NPMLEs Vardi and ELTs Qin by multiplying likelihoods

Truncation

Extreme sample bias with

$$u(x) = \begin{cases} 1, & x \in T \\ 0, & x \notin T \end{cases}$$

Examples

1. Heights of military recruits, above a mininum
2. Swim times of olympic qualifiers, below a maximum
3. Star too dim to be seen

$$L(F) = \prod_{i=1}^{n} \frac{F(\{x_i\})}{\int_{T_i} dF(x)} = \prod_{i=1}^{n} \frac{F(\{x_i\})}{\sum_{j:x_j \in T_i} F(\{x_j\})}$$

Censoring

Instead of exact value, only find that $X_i \in C_i$

$C_i = \{x_i\}$ incorporates uncensored values

Famous example: right censoring of survival time

$$C_i = \begin{cases} \{X_i\}, & X_i \leq Y_i \\ (Y_i, \infty), & X_i > Y_i \end{cases}$$

Censoring vs truncation

Censoring:	Swim times over 3 minutes reported as $(3, \infty)$
Truncation:	Swim times over 3 minutes not reported at all

Coarsening at random

Following truncation to set T_i,

1. Set T_i partitioned into subsets $C_{i,\omega}, \omega \in \Omega_i$
2. X_i is drawn
3. We only learn which C_i contained X_i

Conditional likelihood for censoring

$$L(F) = \prod_{i=1}^{n} \frac{\int_{C_i} dF(x)}{\int_{T_i} dF(x)} = \prod_{i=1}^{n} \frac{\sum_{j:x_j \in C_i} F(\{x_j\})}{\sum_{j:x_j \in T_i} F(\{x_j\})}$$

conditional on the coarsening

More examples

Left truncation:

x_i = brighness of star

y_i = distance

(x_i, y_i) observed $\iff$ $x_i \geq h(y_i)$

Double censoring:

x_i = age when child learns to read

y_i = age when observation ends, right censoring

z_i = age when observation begins, left censoring

Observe $\{x_i\}$ or $[0, z_i)$ or $(y_, \infty]$

Left truncation and right censoring:

As above but only non-readers are observed

Some NPMLEs

Kaplan-Meier for right censored data

$$\widehat{F}((-\infty,t]) = 1 - \prod_{j | t_j \le t} \frac{r_j - d_j}{r_j}$$

$$r_j = \text{Number alive at } t_j-$$

$$d_j = \text{Number dying at } t_j$$

Lynden-Bell (conditional likelihood) for left truncated data

$$\widehat{F}((-\infty,t]) = 1 - \prod_{i=1}^{n} \Big(1 - \frac{1_{x_i \le t}}{\sum_{\ell=1}^{n} 1_{y_\ell < x_i \le x_\ell}}\Big)$$

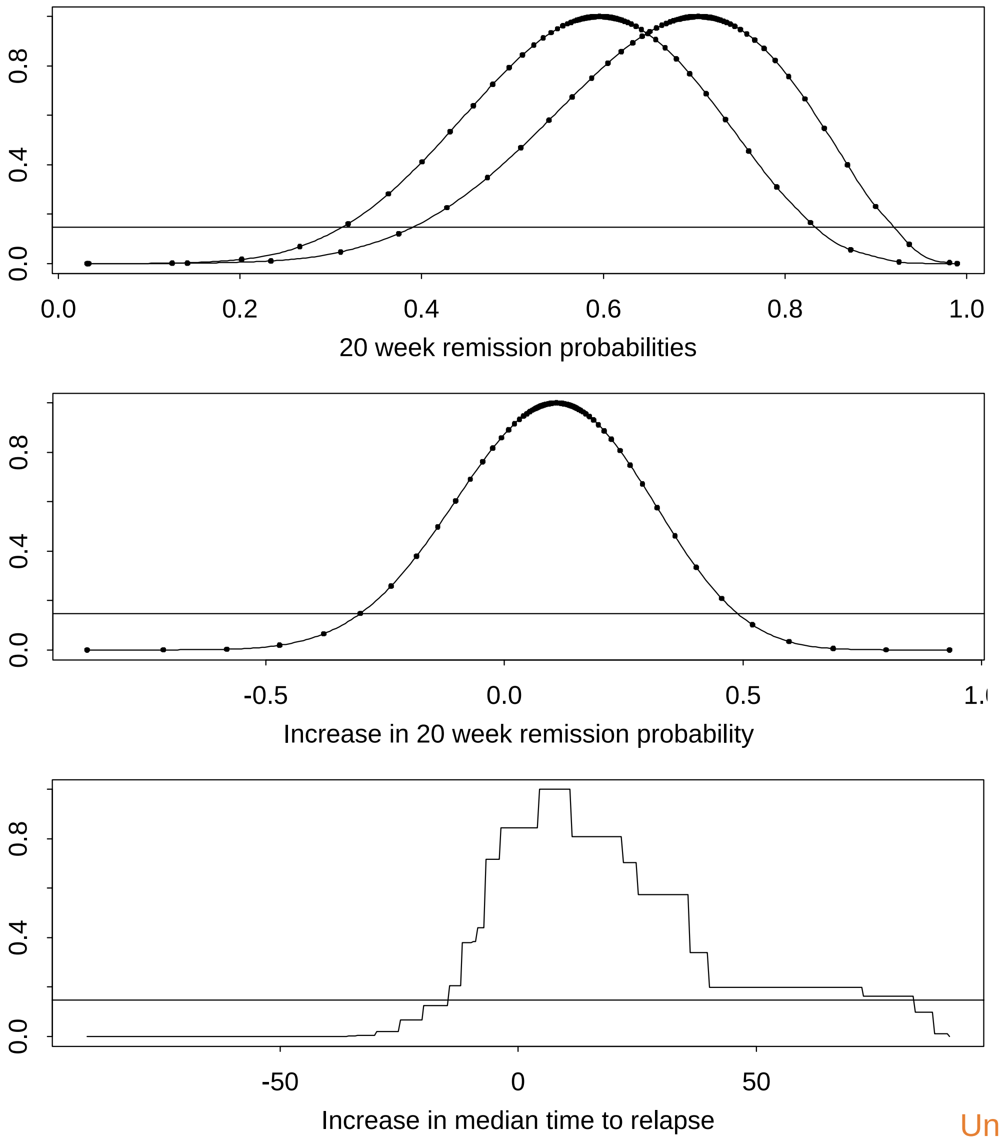
Can have $\widehat{F}((-\infty, x_{(i)}] = 1$ for some $i < n$

Some ELTs

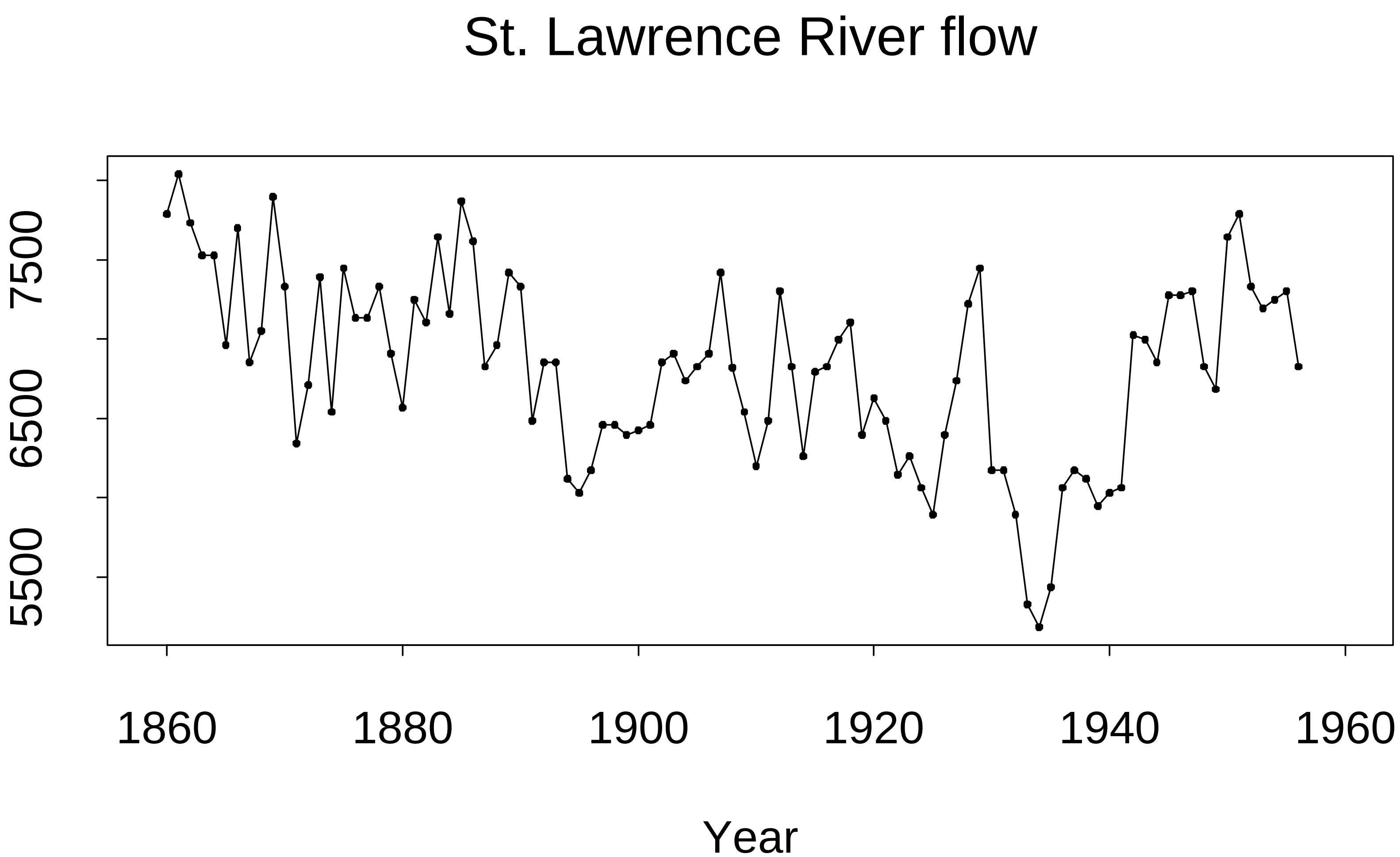
Data type	Statistic	Reference
Right censoring	Survival prob	Thomas & Grunkemeier, Li, Murphy
Left truncation	Survival prob	Li
Left trunc, right cens	Mean	Murphy & van der Vaart
Right censoring	proportional hazard param	Murphy & van der Vaart
Right censoring	integral vs cum hazard	Pan & Zhou

Acute myelogenous leukemia (AML)

Embury et al. Weeks until relapse for 11 with maintainance chemotherapy and 12 non-maintained



Time series



at Ogdensburg Yevjevich

Reduce to independence

$$
\begin{aligned}
Y_i - \mu &= \beta_1(Y_{i-1} - \mu) + \cdots + \beta_k(Y_{i-k} - \mu) + \epsilon_i \\
E(\epsilon_i) &= 0 \\
E(\epsilon_i^2) &= \exp(2\tau) \\
E(\epsilon_i(Y_{i-j} - \mu)) &= 0
\end{aligned}
$$

j	$\hat{\beta}_j$	$-2\log\mathcal{R}(\beta_j = 0)$
1	0.627	30.16
2	-0.093	0.48
3	0.214	4.05

Blocking of time series

Block i of observations, out of $n = \lfloor (T - M)/L + 1 \rfloor$ blocks

$$B_i = \big(Y_{(i-1)L+1}, \ldots, Y_{(i-1)L+M}\big)$$

$$M = \text{length of blocks}$$

$$L = \text{spacing of start points}$$

$$\text{Large } M = L \implies \text{block dependence small}$$

$$\text{Large M} \implies \text{block dependence predictable given } L$$

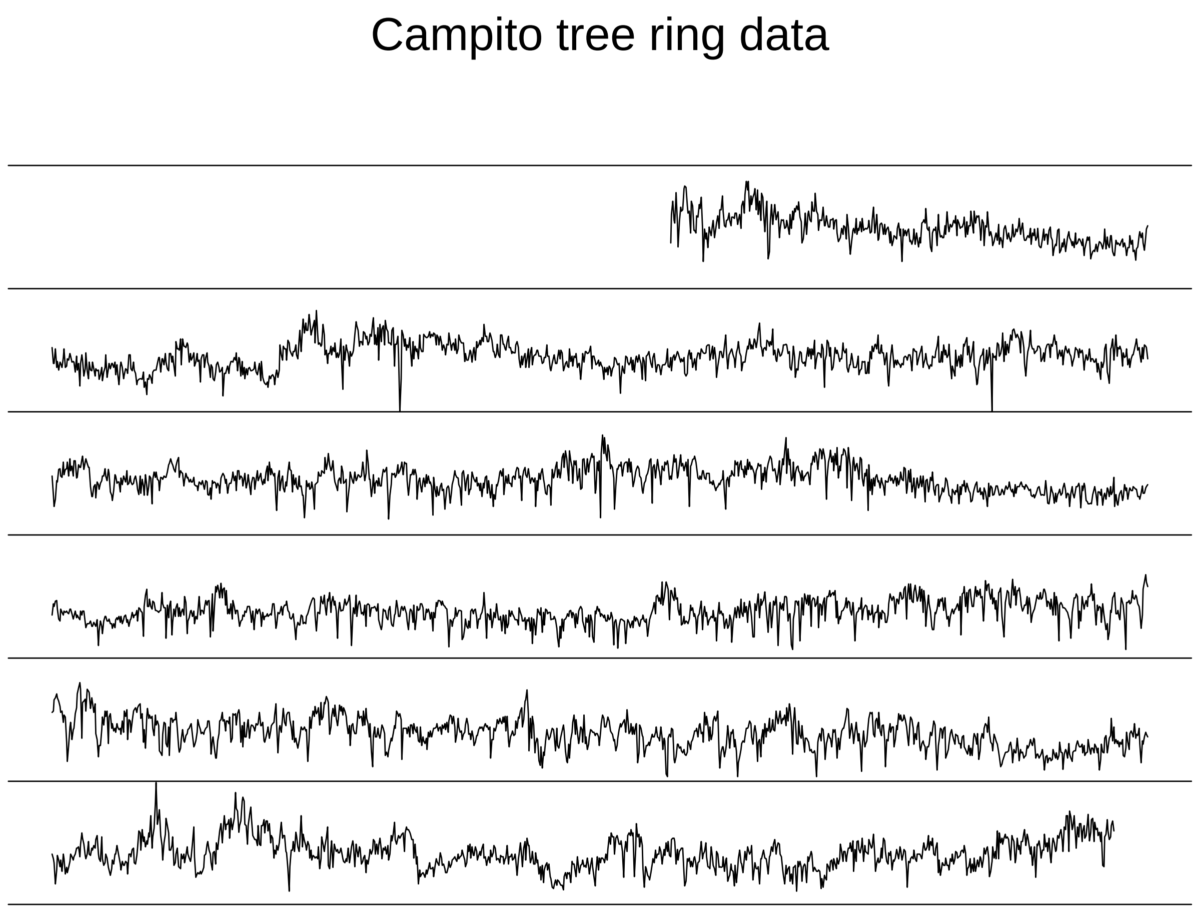
Blocked estimating equation, replace m by b

$$b(B_i, \theta) = \frac{1}{M} \sum_{j=1}^{M} m(X_{(i-1)L+j}, \theta)$$

$$-2\Big(\frac{T}{nM}\Big) \log \mathcal{R}(\theta_0) \to \chi^2 \qquad \text{as } M \to \infty, MT^{-1/2} \to 0$$ Kitamura

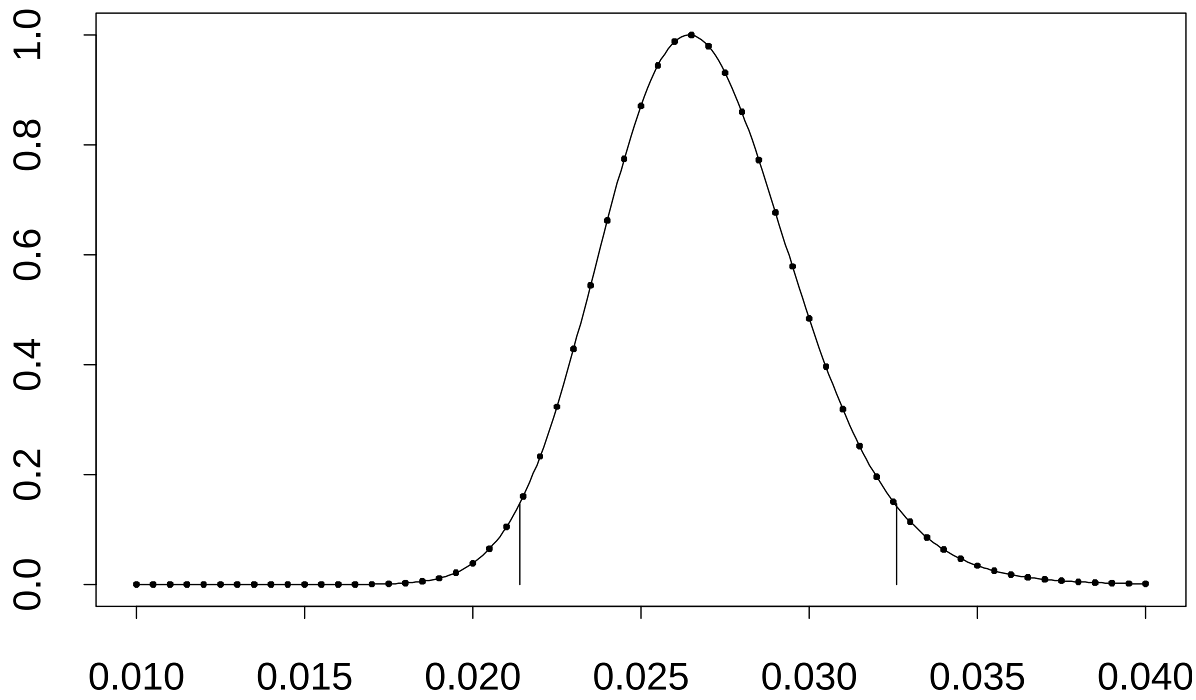
Bristlecone pine

5405 years of Bristlecone pine tree ring widths



0 to 100 in 0.01 mm Fritts et al.

Probability of sharp decrease



Sharp $\equiv$ drop of over 0.2 mm from average of previous 10 years.

MELEs for finite population sampling

1. use side information
   (a) population means, totals, sizes
   (b) stratum means, totals, sizes
2. take unequal sampling probabilities
3. use non-negative observation weights

Hartley & Rao, Chen & Qin, Chen & Sitter

More finite population results

ELTs	$-2\left(1-\frac{n}{N}\right)\mathcal{R}(\mu) \rightarrow \chi^2$	Zhong & Rao
EL variance ests	via pairwise inclusion probabilities	Sitter & Wu
Multiple samples	varying distortions	Zhong, Chen, & Rao

EL hybrids (mostly Jing Qin)

Part of the problem parametric

We want to use that knowledge

Rest of the problem non-parametric

One parametric sample, one not

Y well studied and has parametric distribution

X new and/or does not follow parametric distribution

$$X_i \sim F, \quad i = 1, \ldots, n$$

$$Y_j \sim G(y;\theta), \quad j = 1, \ldots, m$$

$$0 = \int \int h(x, y, \phi) dF(x) dG(y;\theta)$$

e.g. $\phi = E(Y) - E(X)$

Multiply the likelihoods

$$L(F,\theta) = \prod_{i=1}^{n} F(\{x_i\}) \prod_{j=1}^{m} g(y_j;\theta)$$

$$R(F,\theta) = L(F,\theta)/L(\widehat{F},\hat{\theta})$$

$$\mathcal{R}(\phi) = \max_{F,\theta} R(F,\theta) \quad \text{such that}$$

$$0 = \sum_{i=1}^{n} w_i \int h(x_i, y, \phi)dG(y;\theta)$$

Qin gets an ELT

Parametric model for data ranges

$$X \sim \begin{cases} f(x;\theta) & x \in P_0 \\ \text{???} & x \notin P_0 \end{cases}$$

Examples

- Extreme values with exponential tails on $P_0 = [T, \infty)$
- Normal data on $P_0 = [-T, T]$ with outliers

$$L = \prod_{i=1}^{n} f(x_i;\theta)^{x_i \in P_0} \, w_i^{x_i \notin P_0}$$

Define $\mathcal{R}$ using

$$1 = \int_{P_0} dF(x;\theta) + \sum_{i=1}^{n} w_i 1_{x \notin P_0}$$

Qin & Wong get an ELT for means

More hybrids

Parametric	Nonparametric	
$g(y \mid x; \theta)$	$X \sim F$	
$x \sim f(x; \theta)$	$y \mid x \sim G_x$	Few x vals
$x \sim f(x; \theta)$	$(y - \mu(x))/\sigma(x) \sim G$	

Bayesian empirical likelihood (Lazar)

Prior $\theta \sim \pi(\theta)$

$x \sim F$ nonparametric

Posterior $\propto \pi(\theta)\mathcal{R}(\theta)$

Here we have informative prior nonparametric likelihood

Reverse of common practice

Posterior regions asymptotically properly calibrated

Justify via least favorable families

Curve estimation problems

$$\widehat{f}_h(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right) \quad \text{density}$$

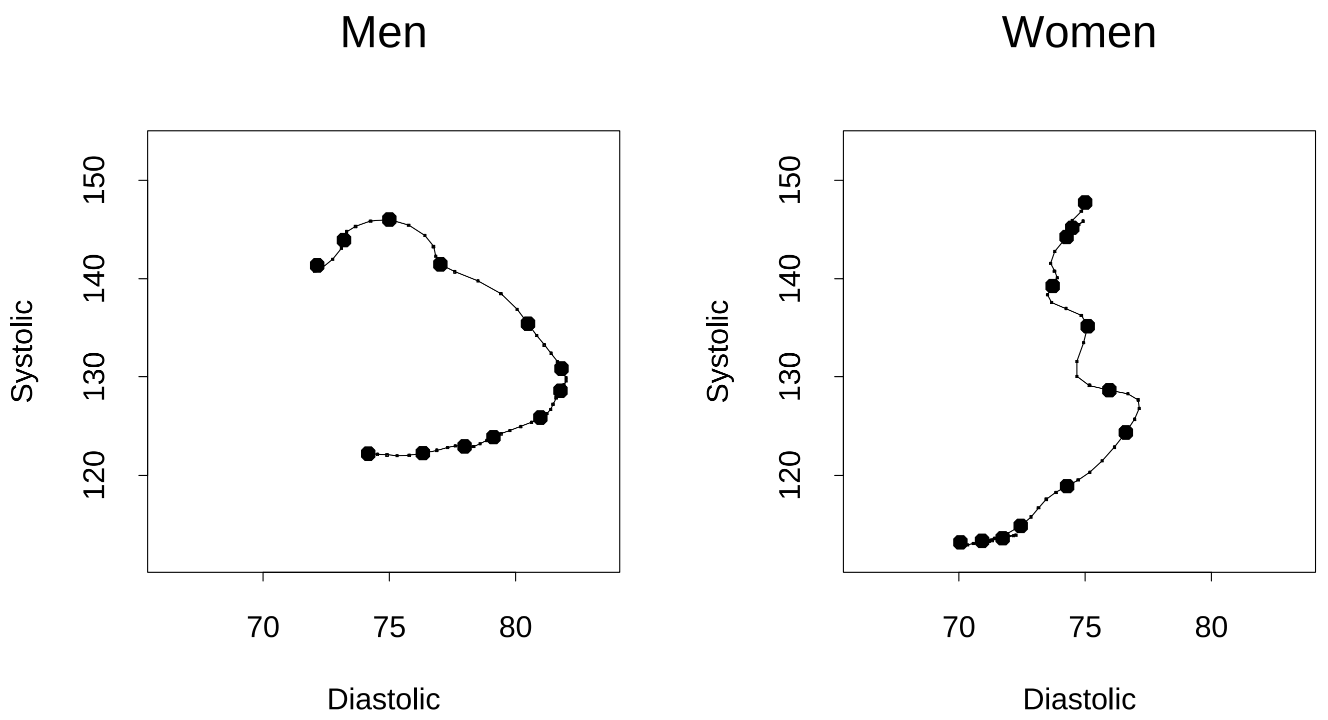
$$\widehat{\mu}_h(x) = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right) Y_i \quad \text{regression}$$

Triangular array ELT applies   Bias adjustment issues

Dimensions and geometry

Dim(x)	Dim(y)	Estimate	Region
1	≥ 2	space curve	confidence tube
≥ 2	1	(hyper)-surface	confidence sandwich

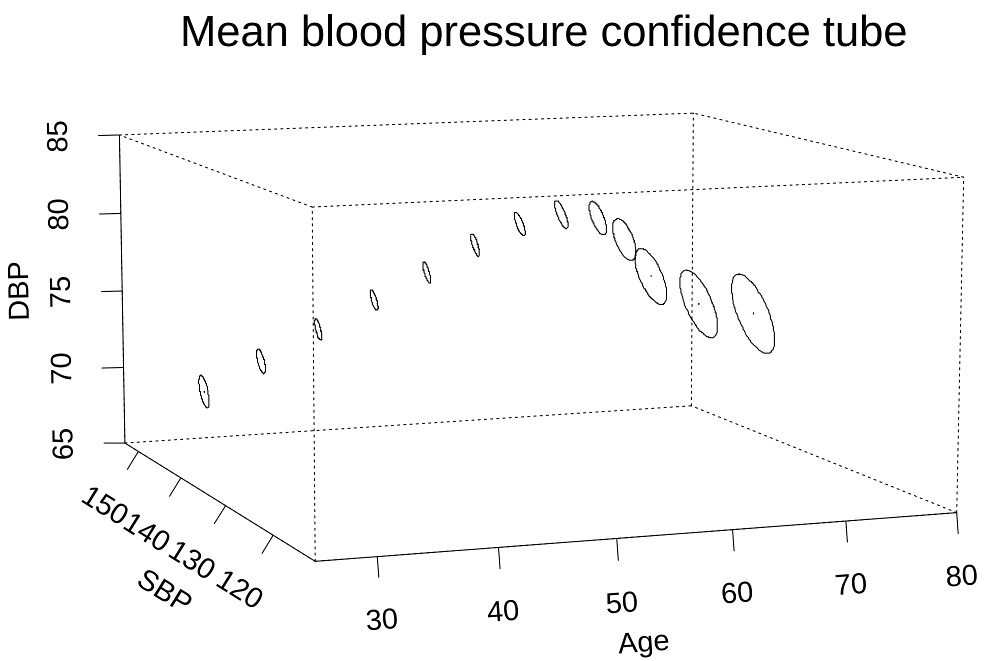
Trajectories of mean blood pressure



dots at ages $25, 30, \ldots, 80$

data from Jackson et al., courtesy of Yee

Confidence tube for men’s mean SBP, DBP



Empirical likelihood vs bootstrap

1. EL gives shape of regions for $d > 1$
2. EL Bartlett correctable, bootstrap not
3. EL can be faster, but,
4. EL optimization can be hard

Why use anything else?

1. Computation is hard
2. Convex hull is binding

Computation

$$\begin{aligned}\log \mathcal{R}(\theta) &= \max_{\nu} \log \mathcal{R}(\theta, \nu) \\ &= \max_{\nu} \min_{\lambda} \mathbb{L}(\theta, \nu, \lambda), \quad \text{where,}\end{aligned}$$

$$\mathbb{L}(\theta, \nu, \lambda) = -\sum_{i=1}^{n} \log\bigl(1 + \lambda' m(x_i, \theta, \nu)\bigr)$$

Inner and outer optimizations $\ll n$ dimensional

Used NPSOL, expensive and not public domain (but it works)

Convex hull

confidence regions nested inside convex hull of data

restrictive if d not small

not so bad for one and two dimensional subparameters

possible remedies

1. Empirical likelihood t Baggerley
2. Hybrid with Euclidean likelihood