Stat 315c: Transposable Data Correspondence Analysis

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Stanford Statistics

Correspondence Analysis

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- This is an old and classical statistical technique pioneered by Jean-Paul Benzécri in the 1960s.
- The treatment by Greenacre is particularly clear.

Contingency tables

$I \times J$ table of counts

$$\begin{matrix} n_{11} & n_{12} & \cdots & n_{1J} \\ n_{21} & n_{22} & \cdots & n_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ n_{I1} & n_{I2} & \cdots & n_{IJ} \end{matrix}$$

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Nomenclature

Correspondence matrix P	$p_{ij} = n_{ij}/n_{\bullet\bullet}$
Row masses	$r_i = p_{i\bullet} = n_{i\bullet}/n_{\bullet\bullet}$
Column masses	$c_j = p_{i\bullet} = n_{i\bullet}/n_{\bullet\bullet}$
Row profiles	$\overline{r}_i = (p_{i1}/r_i, \ldots, p_{iJ}/r_i)' \in \mathbb{R}^J$
Column profiles	$\overline{c}_j = (p_{1j}/c_j, \ldots, p_{Ij}/c_j)' \in \mathbb{R}^I$

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These are conditional and marginal distributions

First moments: centroids

Row centroid

$$\sum_{i=1}^{I} r_i \overline{r}_i = \sum_{i=1}^{I} r_i \left(\frac{p_{i1}}{r_i}, \ldots, \frac{p_{iJ}}{r_i} \right)' = (c_1, \ldots, c_J)' \equiv c$$

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Upshot

'Mass' weighted average of row profiles is marginal distribution over columns

Second moments: inertias

Chisquare for independence as weighted Euclidean distance

$$
\begin{aligned}
X^2 &= \sum_i \sum_j \frac{(n_{ij} - n_{i\bullet} n_{\bullet j}/n_{\bullet\bullet})^2}{n_{i\bullet} n_{\bullet j}/n_{\bullet\bullet}} \\
&= \sum_i n_{i\bullet} \sum_j \frac{(n_{ij}/n_{i\bullet} - n_{\bullet j}/n_{\bullet\bullet})^2}{n_{\bullet j}/n_{\bullet\bullet}} \\
&= n_{\bullet\bullet} \sum_i \frac{n_{i\bullet}}{n_{\bullet\bullet}} \sum_j \frac{(n_{ij}/n_{i\bullet} - n_{\bullet j}/n_{\bullet\bullet})^2}{n_{\bullet j}/n_{\bullet\bullet}} \\
&= n_{\bullet\bullet} \sum_i r_i (\overline{r}_i - c)' \mathrm{diag}(c)^{-1} (\overline{r}_i - c) \\
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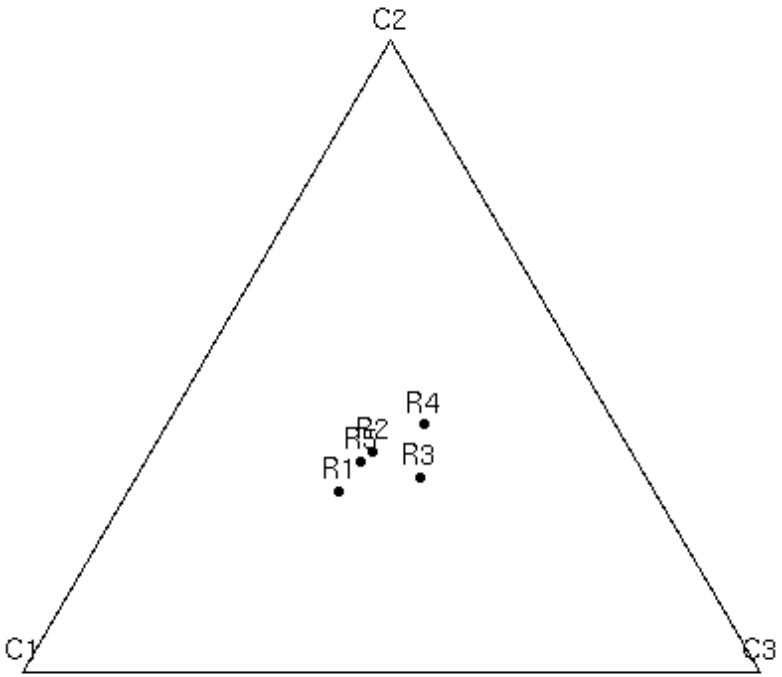
$$

This is the total inertia of the row profiles. It equals total inertia of column profiles.

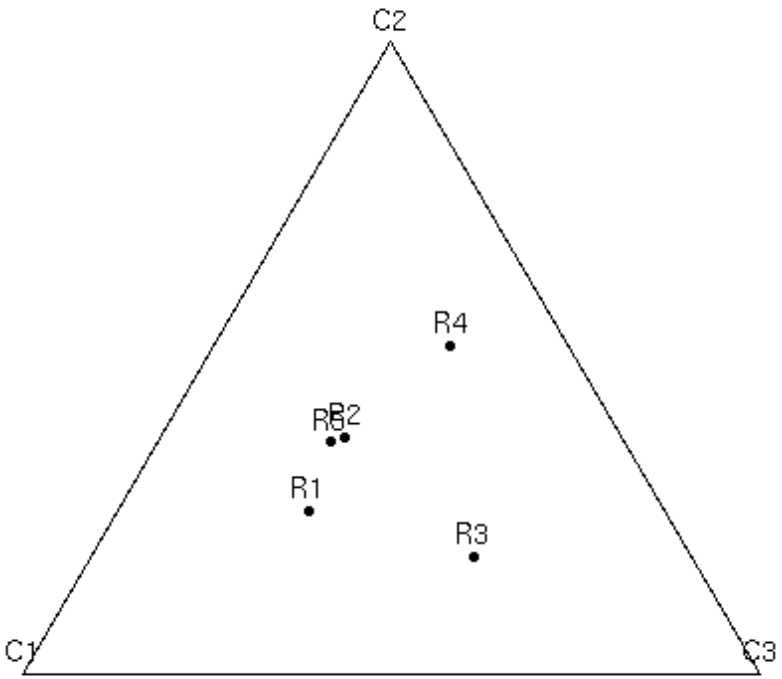
Geometry



For $J = 3$

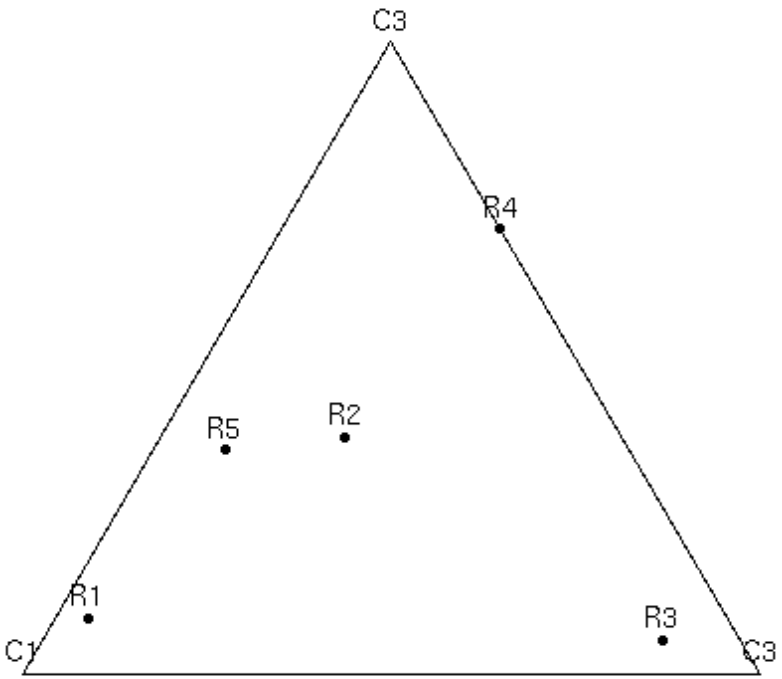
12 8 8
7 7 6
8 8 10
6 9 8
9 8 7
We can plot profiles in $\mathbb{R}^3$
Low inertia

Geometry



This example has higher inertia

Geometry



- Still higher inertia.
- χ^2 statistics describes variation of row profiles
- Similarly for col profiles

Rescale

Distances

- Euclidean distance in plot ignores column values
- Replace $\overline{r}_i$ by $\widetilde{r}_i$ with $\widetilde{r}_{ij} = \dfrac{\overline{r}_{ij}}{\sqrt{c_j}}$
- Euclidean dist between $\widetilde{r}_i$ and $\widetilde{r}_{i'}$ is "χ^2 dist" between $\overline{r}_i$ and $\overline{r}_{i'}$.

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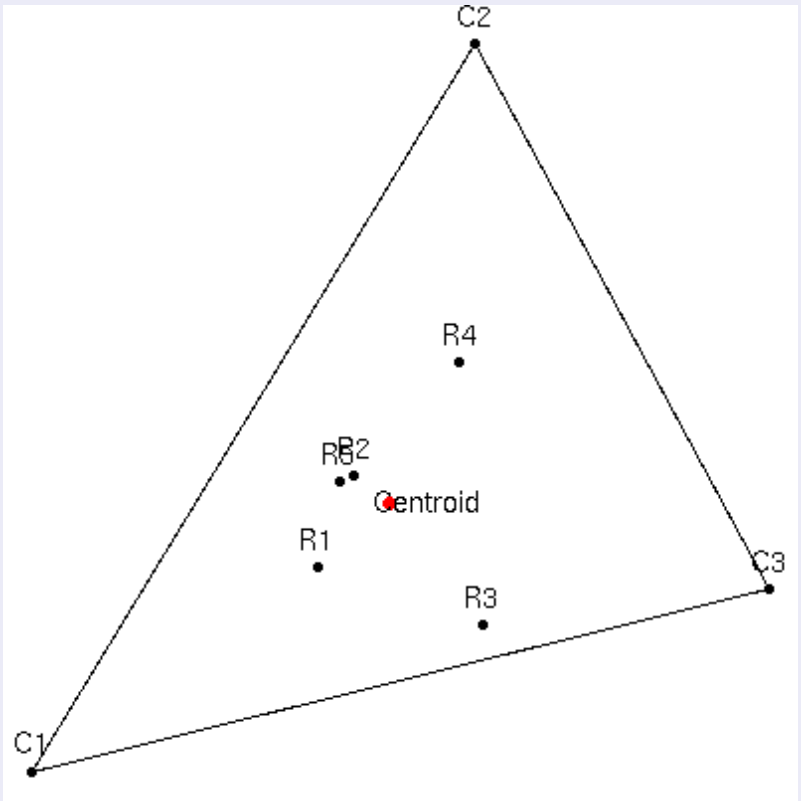

Reason for χ^2

Invariance

- Suppose rows i and i' are proportional
- $n_{ij}/n_{i'j} = \alpha$ all $j = 1, \ldots, J$
- Suppose also that we pool these rows
- New $n_{i^*j} = n_{ij} + n_{i'j}$
- and delete originals

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Then

- New χ^2 distance between cols j and j' equals old dist
- Principle of distributional equivalence
- Common profile, summed mass

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Role of χ^2 in statistical significance is not considered important in this literature

Dimension reduction

Now we have a plot

- With rows and cols both in $\mathbb{R}^{J-1}$

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If J is too big

- reduce dimension
- by principal components of

$$\frac{\overline{r}_{ij} - c_j}{\sqrt{c_j}}$$

- plot in reduced dimension
- along with images of corners

Duality

- Rows lie in $\min(I-1, J-1)$ dimensional space
- So do columns
- In PC of row profiles ... columns are outside
- In PC of column profiles ... rows are outside
- Symmetric correspondence analysis overlap the points after rescaling

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More notation

$$D_r = \text{diag}(r) = \text{diag}(r_1, \ldots, r_I)$$
$$D_c = \text{diag}(c) = \text{diag}(c_1, \ldots, c_J)$$

Symmetric analysis

- Uses SVD $S = U\Sigma V'$ where

$$s_{ij} = \frac{p_{ij} - r_i c_j}{\sqrt{r_i c_j}}$$

- Total inertia is $\|S\|_F^2$
- 'principal inertias' are λ_j^2

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Coordinates

- Rows (1st k cols of)

$$F = (D_r^{-1}P - \mathbf{1}c')D_c^{-1}V_{1:k} = D_r^{-1}U\Sigma$$

- Columns (1st k cols of)

$$G = (D_c^{-1}P - \mathbf{1}r')D_r^{-1}U_{1:k} = D_c^{-1}V\Sigma'$$

Symmetric analysis

Interpretation is tricky/controversial

- r_i near $r_{i'}$ $\surd$
- c_j near $c_{j'}$ $\surd$
- r_i near c_j ??
  Rows and columns are not in the same space

Biplots

- Due to Gabriel (1971) Biometrika
- For matrix X_{ij}
- plot rows as $u_i \in \mathbb{R}^2$
- cols as $v_j \in \mathbb{R}^2$
- with $u_i' v_j \doteq X_{ij}$

A biplot interpretation applies to asymmetric plots

Some finer points

Ghost points

- Apply projection to point not in table
- E.G. hypothetical row entity,
  1. impute a president's 'senate voting record'
  2. compare a state's economy to those of countries
- Treat as fixed profile with mass $\downarrow 0$

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Merged points

- Add linear combination or sum of rows, E.G.
  1. pool columns for math and statistics into "math sciences"
  2. pool rows for EU countries into an EU point

Data types

- Counts are straightforward
- Other 'near measures' are reasonable
  - rainfalls, heights, volumes, temperatures Kelvin
  - dollars spent
  - parts per million
- Reweight cols to equalize inertia $\approx$ standardizing to equalize variance
  Requires iteration

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Fuzzy coding

$x \in \mathbb{R}$ becomes two columns

- $(1, 0)$ for small x say $x < L$
- $(0, 1)$ for large x say $x > U$
- $(1 - t, t)$ for intermediate x $t = (x - L)/(U - L)$

Generalizations to > 2 columns

Puzzlers

- Does it scale? (eg 10^8 points in the plane)
- Is there a tensor version? (Beyond all pairs of two way versions)
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Further reading

- "Correspondence Analysis in Practice" M.J. Greenacre, 1993
  Emphasizes geometry with examples
- "Theory and Applications of Correspondence Analysis" M.J. Greenacre, 1984
  Good coverage of theory with examples
- "Correspondence Analysis and Data Coding with Java and R" F. Murtagh, 2005
  Code and worked examples